**Name:** ______________________________ Quiz 11

Math F252X-901, Calculus II Fall 2024

Thirty minutes maximum. No aids (book, notes, calculator, phone, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably simplified form.

1. Let $f(x) = \sqrt[3]{x}$.

   (a) (8 points.) Find the first order and second order Taylor polynomials for $f(x)$ centered at $a = 8$.

| $n$ | $f'(x)$ | $\frac{f'(8)}{n!}$ |
|---|---|---|
| $0$ | $\sqrt[3]{x}$ | $2$ |
| $1$ | $\frac{1}{3}x^{-2/3}$ | $\frac{1}{12}$ |
| $2$ | $-\frac{2}{9}x^{-5/3}$ | $-\frac{1}{9\cdot 2^4} = -\frac{1}{144}$ |

$$p_1(x) = 2 + \frac{1}{12}(x-8)$$

$$p_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{144}(x-8)^2$$

   (b) (8 points.) Use the first order Taylor polynomial to estimate $\sqrt[3]{7}$.

$$\sqrt{7} \approx p_1(7) = 2 + \frac{1}{12}(7-8)$$

$$= 2 - \frac{1}{12}$$

$$= \frac{23}{12}$$

2. (8 points.) Find the Taylor series for $g(x) = \ln(x)$ centered at $a = 1$ (Wait, haven't we seen this before?).

| $n$ | $f^{(n)}(x)$ | $\frac{f^{(n)}(1)}{n!}$ |
|---|---|---|
| 0 | $\ln(x)$ | 0 |
| 1 | $\frac{1}{x}$ | $\frac{1}{1!}$ |
| 2 | $-1/x^2$ | $-1/2!$ |
| 3 | $2/x^3$ | $2/3!$ |
| 4 | $-3!/x^4$ | $\frac{-3!}{4!}$ |

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$$

3. (8 points.) Find the radius and interval of convergence for the Taylor series you found in problem 2 above.

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^{n+1}(x-1)^n}{n}\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{|x-1|^n}{n}}$$

$$= \lim_{n\to\infty} \frac{|x-1|}{\sqrt[n]{n}} = |x-1|$$ So radius is 1

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by alternating series test

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(0-1)^n}{n} = \sum_{n=1}^{\infty} \frac{-1}{n}$ diverges (harmonic series)

RADIUS OF CONVERGENCE: 1 INTERVAL OF CONVERGENCE: $(0,2]$

BONUS (5 points) What is the 32nd derivative of $f(x) = e^{x^2}$ at $x = 0$. I.E., find $f^{(32)}(0)$. (Hey, this looks familiar too!).

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ so } e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

The $x^{32}$ coefficient here is $\frac{1}{16!}$ (when $n=16$).

In the Maclaurin series for $e^{x^2}$, the $x^{32}$ coefficient is $\frac{f^{(32)}(0)}{32!}$. Since power series are unique, we have

$$\frac{f^{(32)}(0)}{32!} = \frac{1}{16!} \text{ or } f^{(32)}(0) = \frac{32!}{16!}$$