Math 252: Quiz 7 SOLUTIONS **17 March, 2022**

Name: ______________________ ________ / 25

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. [5 points] Suppose $a_n = \dfrac{1}{n}$.

a. Find the limit of the sequence a_n.

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

b. Compute and simplify the first four partial sums $S_1, \ldots, S_4$.

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{3}{2} + \frac{1}{3} = \frac{11}{6}$$

$$S_4 = \frac{11}{6} + \frac{1}{4} = \frac{22+3}{12} = \frac{25}{12}$$

2. [3 points] Find a function $f(n)$ for the nth term a_n of the following recursively defined sequence:

$$a_1 = 3 \text{ and } a_{n+1} = 2a_n \text{ for } n \geq 1.$$

$$a_1 = 3$$

$$a_2 = 3 \cdot 2$$

$$a_3 = 3 \cdot 2^2$$

$$\vdots$$

$$a_n = 3 \cdot 2^{n-1}$$

3. [3 points] Either show that the sequence diverges or, if it converges, find its limit: $a_n = \dfrac{\ln(2n)}{\ln(n^2)}$

$$\lim_{n\to\infty} \frac{\ln(2n)}{\ln(n^2)} = \lim_{n\to\infty} \frac{\ln 2 + \ln n}{2\ln n} \overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{2}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} \qquad \text{(converges)}$$

4. [6 points] State whether the given series converges or diverges, and explain why. If the series converges, find its sum. (**Hint.** Geometric series.)

a. $1 + e + e^2 + e^3 + \ldots$

geometric with $r = e > 1$ $\therefore$ diverges

b. $1 - \dfrac{1}{10} + \dfrac{1}{100} - \dfrac{1}{1000} + \dfrac{1}{10000} - \ldots = \dfrac{1}{1-(-\frac{1}{10})} = \dfrac{1}{\frac{11}{10}} = \dfrac{10}{11}$

geometric with $r = \dfrac{-1}{10}$ so $|r| < 1$,

so converges

5. [8 points] Consider the series
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

a. Use partial fractions and "telescoping" to write a simplified formula for the partial sum $S_k = \sum_{n=1}^{k} \frac{1}{n(n+1)}$.

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \longrightarrow \begin{array}{l} A=1 \\ B=-1 \end{array}$$

So

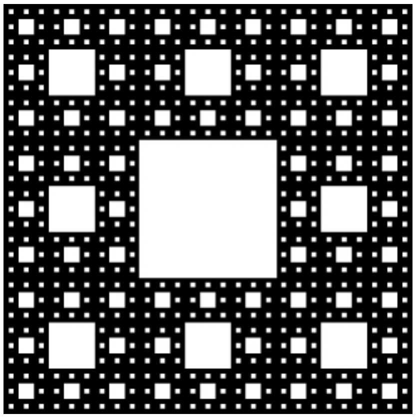
$$S_k = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)}$$
$$= \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\ldots} + \cancel{\frac{1}{k}} - \frac{1}{k+1}$$
$$= 1 - \frac{1}{k+1}$$

b. Compute the value of the series.

So

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} 1 - \frac{1}{k+1}$$
$$= 1 - 0 = 1$$

Extra Credit. [2 points] The black thing below, called the **Sierpinksi gasket**, is built by removing all of the white part from an original fully-black square. Assume the original square has side-length one and thus area one. Remove the middle 1/9th of the area. The remainder is 8 smaller black squares. For each of these, remove the middle 1/9th. Continuing in this way, by infinitely-many stages you remove all the white area. Using geometric series, compute the white area you removed. What area is left, the black area?



$$\begin{pmatrix}\text{white}\\ \text{area}\end{pmatrix} = \frac{1}{9} + 8\cdot\frac{1}{9^2} + 8^2\cdot\frac{1}{9^3} + \ldots$$

is geometric with $a = \frac{1}{9}$, $r = \frac{8}{9} < 1$ <u>so</u>

$$\begin{pmatrix}\text{white}\\ \text{area}\end{pmatrix} = \frac{1/9}{1 - 8/9} = \frac{1/9}{1/9} = 1$$

<u>so</u> $\begin{pmatrix}\text{black}\\ \text{area}\end{pmatrix} = 0.$

[many black points remain, but they take up no area!]

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