Math 252: Quiz 9 **31 March, 2022**

Name: SOLUTIONS

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30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. [9 points] Do the series converge absolutely, conditionally, or neither (diverge)? Show your work and circle one answer.

a. $\sum_{n=1}^{\infty} (-1)^n \frac{n-2}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{n-2}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n} - \frac{2}{\sqrt{n}}}{1} = +\infty$$

so diverges by divergence test

CONVERGES ABSOLUTELY | CONVERGES CONDITIONALLY | DIVERGES

b. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges } (p=2)$$

CONVERGES ABSOLUTELY | CONVERGES CONDITIONALLY | DIVERGES

c. $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{\sqrt{n}}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

$b_n = \frac{1}{\sqrt{n}} \geq 0$

$b_n \to 0$, b_n decreasing

so converges by AST

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges } (p=0.5)$$

CONVERGES ABSOLUTELY | CONVERGES CONDITIONALLY | DIVERGES

2. [6 points] Use the ratio or root test to determine whether the series converges or diverges. Show your work.

a. $\sum_{k=1}^{\infty} \frac{k^3}{3^k}$

$$\rho = \lim_{k\to\infty} \sqrt[k]{\frac{k^3}{3^k}} = \lim_{k\to\infty} \frac{(\sqrt[k]{k})^3}{3} = \frac{1}{3} < 1$$

so converges by root test

[ratio test also works fine]

b. $\sum_{n=1}^{\infty} \frac{(n+2)^2}{n!}$

$$\rho = \lim_{n\to\infty} \frac{\frac{(n+3)^2}{(n+1)!}}{\frac{(n+2)^2}{n!}} = \lim_{n\to\infty} \frac{(n+3)^2 n!}{(n+2)^2 (n+1)!}$$

$$= \lim_{n\to\infty} \frac{(n+3)^2}{(n+2)^2(n+1)} = 0$$ so converge by ratio test

3. [3 points] How close is the partial sum $S_{10} = \sum_{n=1}^{10} \frac{(-1)^n}{2n+1}$ to the convergent infinite sum (series) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$? That is, how large is the remainder R_{10}? Give a brief explanation, and then answer quantitatively in the box, using the fact that the series is alternating.

$b_n = \frac{1}{2n+1}$,

$|R_N| \le b_{N+1}$ } for alt. series

use $N=10$:

$$|R_{10}| \le \frac{1}{2(10+1)+1}$$

$|R_{10}| \le \boxed{\frac{1}{23}}$

4. [4 points] Find the radius and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$. Show your work.

root test:

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(2x)^n}{n}\right|} = \lim_{n\to\infty} \frac{|2x|}{\sqrt[n]{n}} = \frac{|2x|}{1}$$

so $|2x| < 1 \iff -1 < 2x < 1 \iff -\frac{1}{2} < x < \frac{1}{2}$

$x = -\frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges (AST)

$x = \frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic)

$R = \boxed{\frac{1}{2}}$ interval: $\boxed{[-\frac{1}{2}, \frac{1}{2})}$

5. [3 points] Use the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ to find a power series for the function $f(x) = \frac{x^2}{1+x^2}$.

$$f(x) = x^2 \frac{1}{1-(-x^2)} = x^2 \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n+2}$$

Extra Credit. [2 points] Explain why $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$ converges conditionally.

original series converges:

$b_n = \sin\left(\frac{1}{n}\right)$, $b_n \geq 0$, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \sin\left(\frac{1}{n}\right)$
$= \sin(0) = 0$,

b_n decreasing because sin increasing

abs. series diverges:

$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) \leftarrow$ limit comparison:

$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic)

so converges conditionally

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