Math 252 (Bueler): Quiz 10 **18 April 2024**

Name: SOLUTIONS

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30 minutes. No aids (book, notes, calculator, internet, etc.) are permitted. Show all work and use proper notation for full credit. Put answers in reasonably-simplified form. 25 points possible.

1. [6 points] Let $f(x) = \sqrt[3]{x}$.

a. Find the first and second Taylor polynomials, of degrees 1 and 2, of $f(x)$ at basepoint $a = 8$.

$f(x) = x^{1/3}$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f''(x) = \frac{-2}{9}x^{-5/3}$

$p_1(x) = f(a) + f'(a)(x-a)$

$= 2 + \frac{1}{3\cdot 2^2}(x-8)$

$= 2 + \frac{1}{12}(x-8)$

$p_2(x) = 2 + \frac{1}{12}(x-8) - \frac{2}{9\cdot 2^5}(x-8)^2$

b. Use the first Taylor polynomial to estimate $\sqrt[3]{9}$.

$\sqrt[3]{9} = f(9) \approx p_1(9) = 2 + \frac{1}{12}(9-8)$

$= 2 + \frac{1}{12} = \frac{25}{12}$

Calculator:

$\sqrt[3]{9} = 2.0801$

$\frac{25}{12} = 2.0833$

2. [3 points] We know that $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$; this is the 3rd Taylor polynomial at $a = 0$. Evaluate at $-x^2$, and use this to approximate

$$\int_0^1 e^{-x^2}\,dx \approx \int_0^1 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}\,dx$$

$$= \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}\right]_0^1 = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42}$$

$$= \frac{26}{35}$$

Matlab:

$\int_0^1 e^{-x^2}dx = 0.7468$

$26/35 = 0.7429$

3. [8 points] Let $f(x) = \ln(1+x)$.

a. Find the Maclaurin series. (*Any valid method is accepted, including from memory. But get the right series!*)

one way →

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

b. Use the ratio or root test to find the interval of convergence of the same series. (*Hint. Remember to check the endpoints of the interval.*)

$$\lim_{n\to\infty} \frac{\frac{|x|^{n+2}}{n+2}}{\frac{|x|^{n+1}}{n+1}} = \lim_{n\to\infty} \frac{|x|^{n+2}(n+1)}{(n+2)\ |x|^{n+1}} = |x| \lim_{n\to\infty} \frac{n+1}{n+2} = |x|$$

$\therefore |x| < 1 \iff -1 < x < 1$

$\underline{x=-1}$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{n+1}}{n+1} = -\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges (harmonic)

$\underline{x=1}$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges, AST.

$\therefore$ interval: $(-1, 1]$

4. [4 points] Let $f(x) = \sin x$ and $a = 0$, and consider the interval $[-1,1]$. Find the smallest value of n so that the remainder estimate $|R_n(x)| \le \frac{M}{(n+1)!}(x-a)^{n+1}$, where M is an upper bound on $|f^{(n+1)}(z)|$ on the interval, yields $|R_n(x)| \le \frac{1}{20}$ on the interval.

$f^{(n+1)}(z) = \pm \begin{matrix} \sin z \\ \cos z \end{matrix}$ and $\left|\begin{matrix} \sin z \\ \cos z \end{matrix}\right| \le 1$

$-1 \le x \le 1$

$\therefore \quad M = 1$

so:

$|R_n(x)| = \frac{1}{(n+1)!}|x-0|^{n+1} = \frac{|x|^{n+1}}{(n+1)!} \le \frac{1}{(n+1)!}$

so want: $\frac{1}{(n+1)!} \le \frac{1}{20}$

$4! = 24$

so $\frac{1}{(3+1)!} \le \frac{1}{20}$

$\therefore \; n = 3$

5. [4 points] Find the Taylor series for $f(x) = x^2$ around $a = 1$.

$f(x) = x^2$ $\quad f(a) = 1$

$f'(x) = 2x$ $\quad f'(a) = 2$

$f''(x) = 2$ $\quad f''(a) = 2$

$f'''(x) = 0$ $\quad \vdots$

$\vdots$ $\quad f^{(n)}(a) = 0$

$f^{(n)}(x) = 0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= 1 + 2\cdot(x-1) + \frac{2}{2}(x-1)^2 + 0$$

$$= 1 + 2(x-1) + (x-1)^2$$

$$= 1 + 2x - 2 + x^2 - 2x + 1$$

$$= x^2$$

Extra Credit. [2 points] Suppose f is this fifth degree polynomial: $f(x) = 1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5$. Write down a **fully simplified** expression for $p_{17}(x)$, the 17th Taylor polynomial of $f(x)$ at basepoint $a = \sqrt{\pi}$. Explain why your answer, which should require only one line to write, can be written down so immediately.

$$p_{17}(x) = 1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5.$$

this is because $p_{17}(x)$ is the 17th degree polynomial which matches $f(a), f'(a), \dots, f^{(17)}(a)$.

but $f(x)$ is already a polynomial which matches itself perfectly, all the way to the 17th derivative in particular

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