Math 252 (Bueler): Quiz 8 **28 March 2024**

Name: SOLUTIONS

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30 minutes. No aids (book, notes, calculator, internet, etc.) are permitted. Show all work and use proper notation for full credit. Put answers in reasonably-simplified form. 25 points possible.

1. [6 points] Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

a. Use the integral test to show that this series converges.

$$\int_1^\infty \frac{dx}{x^2+1} = \lim_{t\to\infty} \Big[\arctan x\Big]_1^t = \lim_{t\to\infty} \arctan t - \arctan 1$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} < \infty \quad \text{(converges)}$$

So series converges by integral test

b. Use a comparison test to show this series converges. (*Please state which series you are comparing to, and why it converges.*)

$$n^2+1 \ge n^2$$

$$\frac{1}{n^2+1} \le \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges } (p=2)$$

so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by comparison test

2. [3 points] Does the series $\displaystyle\sum_{n=2}^{\infty} \sin n$ converge or diverge? Explain, and identify any tests you use.

$\lim_{n\to\infty} \sin n$ d.n.e. so diverges by divergence test

3. **[9 points]** Use the comparison test or the limit comparison test to determine whether the following series converge or diverge. (*Please state which comparison test you are using, and what series you are comparing to.*)

a. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$

$\sin^2 n \le 1$

$\dfrac{\sin^2 n}{n^2} \le \dfrac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $(p=2)$

so converges by comparison test

b. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^n - 2^n}$

note: $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^n}$ geometric with $r = 3/5 < 1$ converges

$$\lim_{n\to\infty} \frac{\frac{3^n}{5^n-2^n}}{\frac{3^n}{5^n}} = \lim_{n\to\infty} \frac{5^n}{5^n-2^n}$$

$$= \lim_{n\to\infty} \frac{1}{1-2^n/5^n} = \lim_{n\to\infty} \frac{1}{1-(2/5)^n}$$

$$= \frac{1}{1-0} = 1 = L \underset{\neq\infty}{\neq 0}$$ so converges by Lim. Comp. test

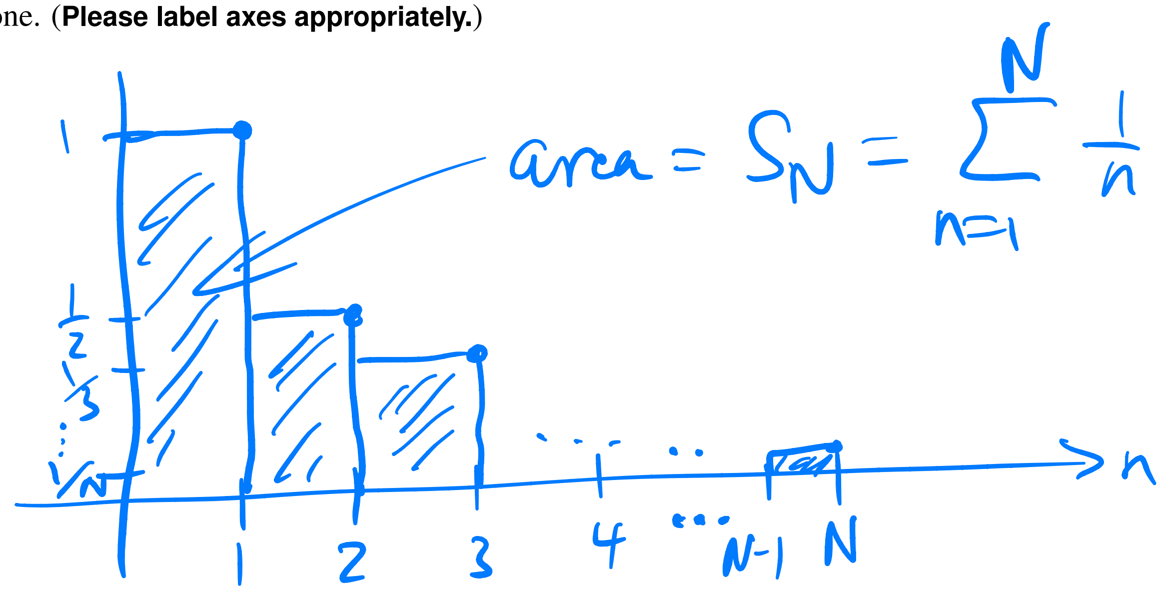
c. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

$\ln n \ge 1$

$\dfrac{\ln n}{n} \ge \dfrac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic)

so $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by comparison test

4. [3 points] Sketch a partial sum S_N of the harmonic series, as the total area of rectangles of width one. (**Please label axes appropriately.**)



5. [2 points] Consider the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p < 0$. Show that such p-series **diverge**. Apply the divergence test.

positive

$$\lim_{n\to\infty} \frac{1}{n^p} = \lim_{n\to\infty} n^{-p} = +\infty \neq 0$$

$\therefore$ diverges

or any "..."

6. [2 points] Simplify the following expression, that is, write it without the factorial: $\frac{n!}{(n+3)!}$

$$\frac{n!}{(n+3)!} = \frac{\cancel{n(n-1)\cdots 2\cdot 1}}{(n+3)(n+2)(n+1)\cancel{n(n-1)\cdots 2\cdot 1}} = \boxed{\frac{1}{(n+3)(n+2)(n+1)}}$$

Extra Credit. [1 point] My computer says that the 20th partial sum of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is $S_{20} = 1.082284588$. How accurate is this as an approximation of the exact infinite sum? Use an integral to estimate the size of the remainder R_{20}.

$$R_N \le \int_N^\infty \frac{1}{x^4}\,dx \underset{\uparrow}{=} \lim_{t\to\infty} \int_{20}^t x^{-4}\,dx$$

$N=20$

$$= \lim_{t\to\infty} \left[\frac{x^{-3}}{-3}\right]_{20}^t = 0 + \frac{1}{3(20)^3} = \frac{1}{24000}$$

$$\approx \frac{1}{20000}$$

$$\approx 0.0005$$ So about 4 or 5 digits are accurate

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