Name: ______________________________ Quiz 4

Math F252X-902, Calculus II Spring 2025

Graded out of 30 points. No aids (book, notes, calculator, phone, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably simplified form.

1. (8 points.) A 1-meter spring requires 10 joules to stretch the spring to 1.1 meters. How much work would it take to stretch the spring from 1 meter to 1.2 meters? Include units with your answer.

$$10 = \int_0^{0.1} kx\,dx$$

$$10 = \frac{kx^2}{2}\Big|_0^{0.1}$$

$$10 = \frac{k(0.01)}{2}$$

$$k = 2000$$

$$\text{work} = \int_0^{0.2} 2000x\,dx$$

$$= 1000x^2\Big|_0^{0.2}$$

$$= 1000(0.04)$$

$$= 40$$

40 joules

2. (6 points.) Find the mass of a metal rod that is 4 meters long (starting at $x = 2$) and has density function $\rho(x) = \dfrac{5}{x}$ grams per meter. Include units with your answer.

$$\int_2^6 \frac{5}{x}\,dx = 5\ln(x)\Big|_2^6$$

$$= 5\ln(6) - 5\ln(2)$$

3. (8 points.) Find the center of mass of a system with the following point masses. Include units with your answer.

- m_1 is 3kg, placed at $(-3, 4)$
- m_2 is 5kg, placed at $(1, 5)$
- m_3 is 2kg, placed at $(2, -1)$

$M_y = 3(-3) + 5(1) + 2(2) = 0$

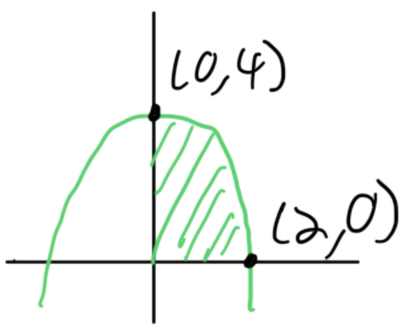
$M_x = 3(4) + 5(5) + 2(-1) = 35$

$m = 3 + 5 + 2 = 10$

ctr of mass: $\left(\frac{0}{10}, \frac{35}{10}\right) = (0, 3.5)$

4. (8 points.) Find the center of mass (or, equivalently, the centroid) for the region in the first quadrant bounded by $y = 4 - x^2$, $y = 0$, and $x = 0$. Simplify ONLY within reason.

$$m = \rho \int_a^b f(x)\, dx, M_x = \rho \int_a^b \frac{(f(x))^2}{2}\, dx, M_y = \rho \int_a^b x f(x)\, dx$$



$m = \int_0^2 4 - x^2\, dx = 4x - \frac{x^3}{3} = 8 - \frac{8}{3} = \frac{16}{3}$

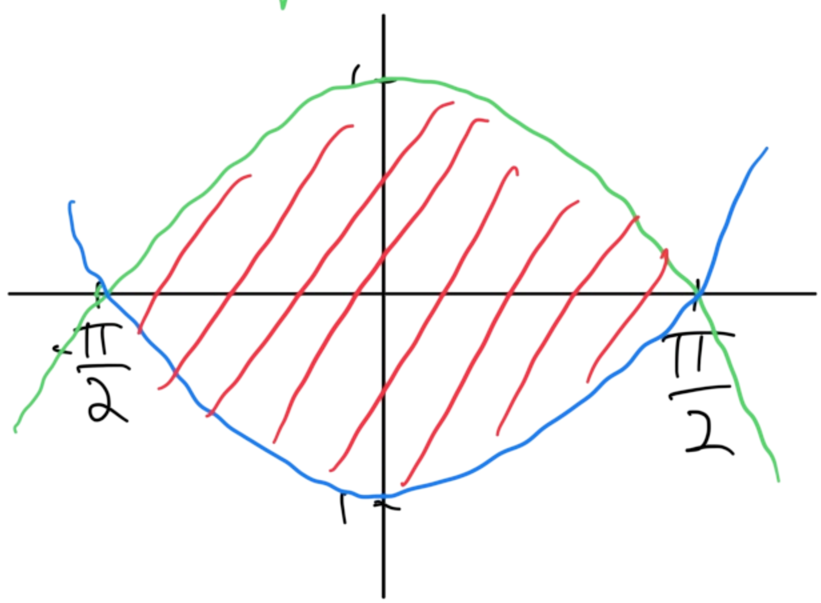
$M_x = \frac{1}{2}\int_0^2 (4-x^2)^2\, dx = \frac{1}{2}\int_0^2 16 - 8x^2 + x^4\, dx = \frac{1}{2}\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_0^2 = \frac{1}{2}\left(32 - \frac{64}{3} + \frac{32}{5}\right)$

$M_y = \int_0^2 x(4-x^2)\, dx = \int_0^2 4x - x^3\, dx = \left[2x^2 - \frac{x^4}{4}\right]_0^2 = 8 - 4 = 4$

Centroid: $\left(\frac{4}{\left(\frac{16}{3}\right)}, \frac{32 - \frac{64}{3} + \frac{32}{5}}{\left(\frac{16}{3}\right)}\right) = \left(\frac{3}{4}, \frac{3\left(32 - \frac{64}{3} + \frac{32}{5}\right)}{16}\right)$

5. (3 points.) (BONUS) Find the centroid (center of mass) for the region bounded by $y = \cos(2x)$ and $y = \sin\left(2x - \frac{\pi}{2}\right)$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Answers without justification will receive zero points.

The graph of $y = \sin(2x - \frac{\pi}{2})$ can be obtained by taking the graph of $y = \sin(x)$, shifting right by $\frac{\pi}{2}$ and compressing horizontally by a factor of 2. The graph of $y = \cos(2x)$ can be obtained by taking the graph of $y = \cos(x)$ and compressing by a factor of 2. This yields



This region is symmetric about the x-axis and about the y-axis, so the centroid is $(0,0)$.