SOLUTIONS

Worksheet 2.4: Length of a curve and area of a surface

Set up, *but do not evaluate*, definite integrals for these length and area problems.

1. Find the length of the curve $y = e^x$ from $x = 0$ to $x = 1$.

$$L = \int_0^1 \sqrt{1 + e^{2x}}\, dx$$

$$f(x) = e^x$$
$$f'(x) = e^x$$

2. Find the surface area of the surface of revolution from rotating $y = e^x$ from $x = 0$ to $x = 1$ around the x-axis.

$$A = \int_0^1 2\pi\, e^x \sqrt{1 + e^{2x}}\, dx$$

3. Find the length of the curve $y = \frac{1}{4}x^{3/2}$ from $x = 0$ to $x = 3$.

$$L = \int_0^3 \sqrt{1 + \left(\tfrac{3}{8}x^{1/2}\right)^2}\, dx$$
$$= \int_0^3 \sqrt{1 + \tfrac{9}{64}x}\, dx$$

$$f(x) = \tfrac{1}{4}x^{3/2}$$
$$f'(x) = \tfrac{3}{8}x^{1/2}$$

4. Find the length of the curve $x = y^3$ from the point $(1, 1)$ to the point $(8, 2)$.

$$L = \int_1^2 \sqrt{1+9y^4}\, dy \qquad g(y) = y^3,\quad g'(y) = 3y^2$$

or

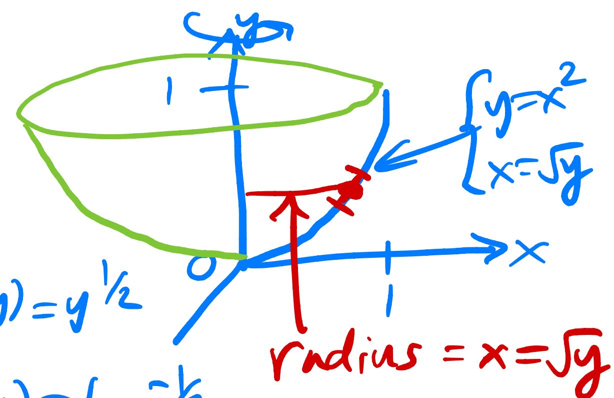
$$L = \int_1^8 \sqrt{1+\tfrac{1}{9}x^{-4/3}}\, dx \qquad f(x) = x^{1/3},\quad f'(x) = \tfrac{1}{3}x^{-2/3}$$

5. Find the surface area of the surface of revolution from rotating $y = x^2$ from $x = 0$ to $x = 1$ around the y-axis.

$$A = \int_0^1 \underbrace{2\pi\sqrt{y}}_{2\pi r}\ \cdot\ \underbrace{\sqrt{1+\left(\tfrac{1}{2\sqrt{y}}\right)^2}\, dy}_{dL}$$

$$= 2\pi \int_0^1 \sqrt{y}\sqrt{1+\frac{1}{4y}}\, dy$$

$$g(y) = y^{1/2},\quad g'(y) = \tfrac{1}{2}y^{-1/2}$$



6. Now do triage. Which of the integrals in problems **1** through **5** can actually be computed by hand? Try those. (I'll give my answer at the board.) For the others, go online and use your favorite tool to compute values for the definite integrals.

1. Don't know how. Wolfram α gives $L \approx 2.0035$

2. With $u = e^x$ get $A = 2\pi \int_1^e \sqrt{1+u^2}\, du$. In section 3.3 we will do this by-hand. Wolfram α gives $A \approx 22.943$

3. Do it:
$$L = \int_0^3 \sqrt{1+\tfrac{9}{64}x}\, dx = \int_1^{91/64} \sqrt{u}\ \frac{64}{9}\, du \qquad \left[\begin{array}{l} u = 1+\frac{9}{64}x \\ \frac{64}{9}du = dx \end{array}\right]$$
$$= \frac{64}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{91/64} = \frac{128}{27}\left(\left(\frac{91}{64}\right)^{3/2} - 1\right)$$

4. Don't know how. Wolfram α gives $L \approx 7.08246$ for both

5. Do it: $A = 2\pi \int_0^1 \sqrt{y+\tfrac{1}{4}}\, dy \underset{u=y+\frac{1}{4}}{=} \cdots = \frac{4\pi}{3}\left(\left(\frac{5}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right)$.