SOLUTIONS

 Not to be turned in!

Worksheet: Improper integrals

Compute these integrals with friends! Also, please carefully write the limit, for example

$$\int_1^\infty \frac{dx}{x^2} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^2} = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty} 1 - \frac{1}{t} = 1$$

A. $\int_2^\infty \frac{1}{9+x^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{3^2+x^2}\,dx = \lim_{t\to\infty} \int_{\arctan(2/3)}^{\arctan(t/3)} \frac{3\sec^2\theta\,d\theta}{3^2\sec^2\theta}$

$3\tan\theta = x$

$= \frac{1}{3}\lim_{t\to\infty}\int_{\operatorname{atan}(2/3)}^{\operatorname{atan}(t/3)} d\theta = \frac{1}{3}\lim_{t\to\infty} \operatorname{atan}(t/3) - \operatorname{atan}(2/3)$

$= \frac{1}{3}(\pi/2 - \operatorname{atan}(2/3))$

converges

B. $\int_{-\infty}^0 e^x\,dx = \lim_{t\to-\infty}\int_t^0 e^x\,dx = \lim_{t\to-\infty}\left[e^x\right]_t^0$

$= \lim_{t\to-\infty} e^0 - e^t = \lim_{t\to-\infty} 1 - e^t = 1 - 0 = 1$

converges

C. $\int_0^1 \frac{1}{\sqrt[4]{x}}\,dx = \lim_{t\to0^+}\int_t^1 x^{-1/4}\,dx = \lim_{t\to0^+}\left[\frac{4}{3}x^{3/4}\right]_t^1$

$= \lim_{t\to0^+}\frac{4}{3}(1 - t^{3/4}) = \frac{4}{3}(1-0) = \frac{4}{3}$

converges

D. $\int_0^1 \ln t\,dt = \lim_{a\to0^+}\int_a^1 \ln t\,dt = \lim_{a\to0^+}\left(\left[t\ln t\right]_a^1 - \int_a^1 \frac{1}{t}t\,dt\right)$

$\left(\begin{array}{ll} u=\ln t & v=t \\ du=\frac{1}{t} & dv=dt\end{array}\right)$

$= \lim_{a\to0^+}\left(1\cdot 0 - a\cdot\ln a - \int_a^1 dt\right) = \lim_{a\to0^+} -a\ln a - (1-a)$

how to do this? L'Hopital's rule!

$= 0 - 1 + 0 = -1$

converges

E. $\int_1^2 \frac{dx}{1-x} = \lim_{t\to 1^+} \int_t^2 \frac{dx}{1-x} = \lim_{t\to 1^+}\left[-\ln|1-x|\right]_t^2$

diverges

$$= \lim_{t\to 1^+} -\ln 1 + \ln|1-t| = \lim_{t\to 1^+} \ln(t-1) = -\infty$$

diverges

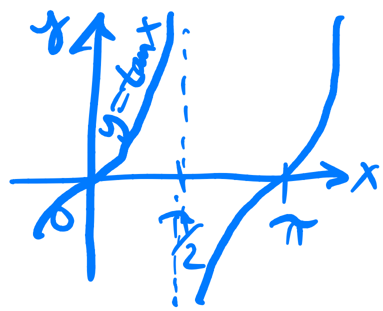
F. $\int_0^\infty e^x e^{-sx}\, dx = \lim_{t\to\infty} \int_0^t e^{(1-s)x} dx = \lim_{t\to\infty}\left[\frac{e^{(1-s)x}}{1-s}\right]_0^t$

$(s>1)$

converges if $s>1$ (otherwise diverges)

$$= \frac{1}{1-s}\lim_{t\to\infty}\left(e^{(1-s)t} - e^0\right) = \frac{1}{1-s}(0-1) = \frac{1}{s-1}$$

G. $\int_0^\pi \tan x\, dx = \int_0^{\pi/2} \tan x\, dx + \int_{\pi/2}^{\pi} \tan x\, dx$



diverges

$$\int_0^{\pi/2} \tan x\, dx = \lim_{t\to \frac{\pi}{2}^-} \int_0^t \frac{\sin x}{\cos x} dx$$

$$= \lim_{t\to \frac{\pi}{2}^-}\left[-\ln|\cos x|\right]_0^t = \lim_{t\to \frac{\pi}{2}^-}\left(-\ln|\cos t| + \underbrace{\ln 1}_{0}\right) = +\infty$$

diverges

H. $\int_2^\infty \frac{dx}{x \ln^3 x} = \lim_{t\to\infty} \int_2^t \frac{dx}{x(\ln x)^3}$

converges

$$= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{du}{u^3} = \lim_{t\to\infty}\left[-\frac{u^{-2}}{2}\right]_{\ln 2}^{\ln t}$$

$(u = \ln x)$

$$= \lim_{t\to\infty}\left(\frac{1}{2(\ln 2)^2} - \frac{1}{2(\ln t)^2}\right) = \frac{1}{2(\ln 2)^2} - 0 = \frac{1}{2(\ln 2)^2}$$