SOLUTIONS

Worksheet: Does the series converge or diverge?

For each of the following 13 infinite series, state whether it converges or diverges. Justify your statement using the following tests (or known series):

- geometric series
- telescoping series
- p-series
- divergence test
- integral test
- comparison test
- limit comparison test

In many cases multiple tests can determine convergence or divergence.

A. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}$ Converges compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ (geometric, $r=\frac{1}{2}<1$)

$\frac{1}{n2^n} \le \frac{1}{2^n}$ or limit compare

B. $\displaystyle\sum_{n=1}^{\infty} 2^n$ diverges divergence test: $\lim_{n\to\infty} 2^n = +\infty \ne 0$

or geometric with $r=2>1$

C. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ Converges limit compare to (for example) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3/2)^n}$, which is geometric with $r=2/3<1$:

or integral test (but hard)

$\displaystyle\lim_{n\to\infty} \frac{n/2^n}{(2/3)^n} = \lim_{n\to\infty} \frac{n}{(4/3)^n} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{(4/3)^n \ln(4/3)} = 0$

D. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ Converges integral test: $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^3} = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{du}{u^3}$ $[u=\ln x]$

$\displaystyle = \lim_{t\to\infty} \left[\frac{u^{-2}}{-2}\right]_{\ln 2}^{\ln t} = \lim_{t\to\infty} \frac{1}{2}\left((\ln 2)^{-2} - (\ln t)^{-2}\right) = \frac{1}{2(\ln 2)^2}$

$\ne 0, +\infty$

E. $\displaystyle\sum_{n=1}^{\infty} \frac{n-4}{n^3+2n}$ Converges limit compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ ($p=2$ series):

$\displaystyle\lim_{n\to\infty} \frac{\frac{n-4}{n^3+2n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3-4n}{n^3+2n} = 1$

F. $\displaystyle\sum_{n=2}^{\infty} \frac{1+\cos(n)}{e^n}$ Converges compare to $\displaystyle\underbrace{\sum_{n=2}^{\infty} \frac{2}{e^n}}_{\text{geometric with } r=\frac{1}{e}<1}$ because $1+\cos(n) \le 2$

G. $\displaystyle\sum_{n=3}^{\infty} \frac{n^2}{\sqrt{n^3-1}}$ diverges limit compare to $\displaystyle\sum_{n=3}^{\infty} \frac{n^2}{\sqrt{n^3}} = \sum_{n=3}^{\infty} n^{1/2}$ which diverges

or integral test

$\displaystyle\lim_{n\to\infty} \frac{n^2/\sqrt{n^3-1}}{n^2/\sqrt{n^3}} = \dots = 1$

$\left(\displaystyle\lim_{n\to\infty} n^{1/2} \neq 0\right)$ by divergence test

H. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{(n^4-3)^2}$ converges limit compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$ which converges (p=5 series)

or integral test

$\displaystyle\lim_{n\to\infty} \frac{n^3/(n^4-3)^2}{1/n^5} = \lim_{n\to\infty} \frac{n^8}{(n^4-3)^2} = 1$

I. $\displaystyle\sum_{n=1}^{\infty} (-1)^n 3^{-n/3}$ converges geometric: $(-1)^n 3^{-n/3} = \left(\frac{-1}{\sqrt[3]{3}}\right)^n$ so $|r| = \frac{1}{\sqrt[3]{3}} < 1$

J. $\displaystyle\sum_{n=2}^{\infty} \frac{|\sin(n)|}{n}$

unknown to me!

K. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n!}$ converges compare to (for example) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ ← p=2 series

$n! > n^2$ starting with $n=4$

L. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges integral test: $\displaystyle\int_1^{\infty} \frac{x\,dx}{x^2+1} = \lim_{t\to\infty} \frac{1}{2}\int_2^{t^2+1} \frac{du}{u}$ $[u = x^2+1]$

or limit comparison

$= \frac{1}{2}\displaystyle\lim_{t\to\infty} \ln(t^2+1) - \ln 2 = +\infty$

M. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges limit compare to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ (p=2 series)

or integral

or telescoping

$\displaystyle\lim_{n\to\infty} \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1$

Finally, an observation and a question. In every case above you could use a computer to find s_{1000}, the (partial) sum of the first thousand terms. In which could you find the exact sum of the series?

Answer:
- I is geometric, so you can find the sum. $\left(\text{sum} = \frac{a}{1-r}\right)$
- we will see for K: (sum) $= e - 2$
- sums are possible but obscure, for A and C

it is rare to be able to find sum!