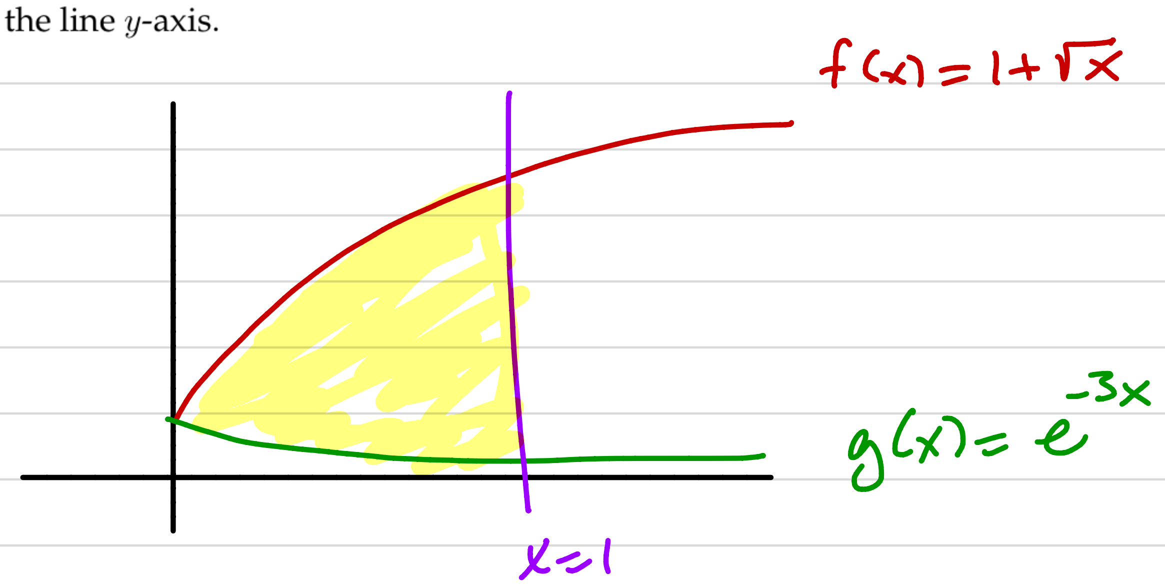
Solutions to Review Problems

1. Let R be the region bounded by the graph of $f(x) = 1 + \sqrt{x}$, $g(x) = e^{-3x}$ and the vertical line $x = 1$. Sketch the region R.

   (a) Set up, but do not solve, an integral that gives the area of R.

   (b) Set up, but do not solve, an integral that finds the volume of the solid when R is rotated about the x-axis.

   (c) Set up, but do not solve, an integral that finds the volume of the solid when R is rotated about the line y-axis.



a. $\displaystyle\int_0^1 \left(1 + \sqrt{x} - e^{-3x}\right) dx$

b. $\displaystyle\pi \int_0^1 (1 + \sqrt{x})^2 - \left(e^{-3x}\right)^2 \, dx$

c. $\displaystyle 2\pi \int_0^1 x\left(1 + \sqrt{x} - e^{-3x}\right) dx$

2. Evaluate the following integrals.

(a) $\int \sin^5(2x)\cos^2(2x)dx$

(b) $\int \frac{2x^2+3x-2}{x^3-x^2}dx$

(c) $\int \tan^{-1}\left(\frac{x}{2}\right)dx$

(d) $\int \frac{x^2}{(4-x^2)^{3/2}}dx$

a. $\int \sin^5(2x)\cos^2(2x)\,dx$

$= \int \sin^4(2x)\cos^2(2x)\sin(2x)\,dx$

$= \int (1-\cos^2(2x))^2\cos^2(2x)\sin(2x)\,dx$

$u=\cos(2x)$

$du=-\sin(2x)\cdot 2\,dx$

$-\frac{1}{2}du = \sin(2x)\,dx$

$= -\frac{1}{2}\int (1-u^2)^2 u^2\,du$

$= -\frac{1}{2}\int u^2-2u^4+u^6\,du$

$= -\frac{1}{2}\left[\frac{1}{3}u^3-\frac{2}{5}u^5+\frac{1}{7}u^7\right]+C$

$= -\frac{1}{6}\cos^3(2x)+\frac{1}{5}\cos^5(2x)-\frac{1}{14}\cos^7(2x)+C$

(b) $\frac{2x^2+3x-2}{x^3-x^2} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}$ or $2x^2+3x-2=Ax(x-1)+B(x-1)+Cx^2$

$x=0:\ -2=-B,\ B=2$

$x=1:\ 3=C$

$x=-1:\ -3=2A+2(-2)+3,\ A=-1$

So $\int \frac{2x^2+3x-2}{x^3-x^2}dx = \int\left(\frac{-1}{x}+\frac{2}{x^2}+\frac{3}{x-1}\right)dx = -\ln|x|-2x^{-1}+3\ln|x-1|+C$

(c) $\int \arctan\left(\frac{x}{2}\right)dx = x\arctan\left(\frac{x}{2}\right)-\int\frac{2x}{4+x^2}dx$

$u=\arctan\left(\frac{x}{2}\right)$ $\quad dv=dx$

$du=\frac{1}{2}\left(\frac{1}{1+\left(\frac{x}{2}\right)^2}\right)dx$ $\quad v=x$

$=\frac{1}{2}\left(\frac{1}{1+\frac{x^2}{4}}\right)\cdot\frac{4}{4}=\frac{2}{4+x^2}$

$= x\arctan\left(\frac{x}{2}\right) - \ln(4+x^2)+C$

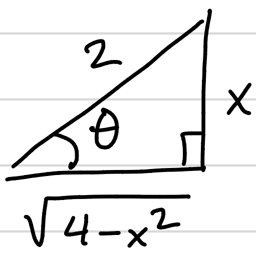
$$\int \frac{x^2\,dx}{(4-x^2)^{3/2}} = \int \frac{4\sin^2\theta \cdot 2\cos\theta\,d\theta}{8\cos^3\theta} = \int \tan^2\theta\,d\theta = \int (\sec^2\theta - 1)\,d\theta$$

$$= \tan\theta - \theta + C$$

$$= \frac{x}{\sqrt{4-x^2}} - \arcsin\left(\frac{x}{2}\right) + C$$

$x = 2\sin\theta \qquad dx = 2\cos\theta\,d\theta$

$$(4-x^2)^{3/2} = (4-4\sin^2\theta)^{3/2} = (4\cos^2\theta)^{3/2} = 8\cos^3\theta$$



3. Let $a_n = \ln\left(\dfrac{2n^2+1}{3n^2+4}\right)$.

(a) Determine whether the sequence a_n converges. If it is convergent determine what it converges to.

(b) Determine whether the series $\sum_{n=1}^{\infty} a_n$ converges or diverges.

a. $\lim_{n\to\infty} \ln\left(\dfrac{2n^2+1}{3n^2+4}\right) = \ln\left(\dfrac{2}{3}\right)$. It converges.

b. $\sum a_n$ diverges by the Divergence test and part ⓐ. Since $\ln\left(\frac{2}{3}\right) \neq 0$.

$n=1$

4. Determine if the series below converge or diverge. Full credit will only be given for answers that include (1 pt) the name of the test being applied, (5 pts) a complete application of the test, including evidence that the conditions have been met, and (1 pt) a clear conclusion with justification.

(a) $\sum_{n=1}^{\infty} \frac{n^2+1}{2n^3+2}$

(b) $\sum_{n=1}^{\infty} \frac{\sin(3n)}{2+n^4}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$

(d) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{3/2}}$

a. diverges by limit comparison test.

Compare to $\sum \frac{1}{n}$, a divergent p-series.

$$\lim_{n\to\infty} \frac{n^2+1}{2n^3+2} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^3+n}{2n^3+2} = \frac{1}{2}$$

b. $\sum_{n=1}^{\infty} \frac{\sin(3n)}{2+n^4}$. It's absolutely convergent using the direct comparison test. Compare to $\sum_{n=1}^{\infty} \frac{1}{n^4}$, a convergent p-series

$$\frac{|\sin(3n)|}{2+n^4} \le \frac{1}{n^4}.$$

c. A.S.T. $b_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$; $\frac{1}{\sqrt{n+2}+\sqrt{n+1}} < \frac{1}{\sqrt{n+1}+\sqrt{n}}$

So b_n's are decreasing.

$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0.$ So the series converges

d. Integral test : $\int_2^{\infty} \frac{(\ln x)^{-3/2}}{x}\,dx = \lim_{n\to\infty} \int_2^{n} \frac{(\ln x)^{-3/2}}{x}\,dx$

$$= \lim_{n\to\infty} -2(\ln x)^{-1/2}\Big|_2^n = \lim_{n\to\infty} -2\left(\frac{1}{\sqrt{\ln n}} - \frac{1}{\sqrt{\ln 2}}\right) = \frac{2}{\sqrt{\ln 2}}$$

Since the integral converges, the series converges.

5. Find the sum of the following series exactly.

a) $\sum_{n=1}^{\infty}(-3)^{n+1}5^{-n}$

b) $\sum_{n=0}^{\infty}\frac{(-1/2)^n}{n!}$

a. $\sum_{n=1}^{\infty} \frac{9}{5} \cdot \left(\frac{-3}{5}\right)^{n-1} = \frac{\frac{9}{5}}{1+\frac{3}{5}} = \frac{9}{5+3} = \frac{9}{8}$

b. $\sum_{n=0}^{\infty} \frac{(-\frac{1}{2})^n}{n!} = e^{-\frac{1}{2}}$