1. Find the Taylor series for the function $f(x) = \sin(x)$ centered at $a = \pi$.

$x = \pi$

$f(x) = \sin(x) = f^{(4)}(x)$ $\quad$ $f(\pi) = 0$

$f'(x) = \cos(x) = f^{(5)}(x)$ $\quad$ $f'(\pi) = -1 = f^{(5)}(\pi)$

$f''(x) = -\sin(x) = f^{(6)}(x)$ $\quad$ $f''(\pi) = 0$

$f'''(x) = -\cos(x) = f^{(7)}(x)$ $\quad$ $f'''(\pi) = 1 = f^{(7)}(\pi)$

$$\sin(x) = \sum_{i=0}^{\infty} \frac{(-1)^{n+1}(x-\pi)^{2n+1}}{(2n+1)!}$$

2. (a) Determine whether the improper integral $\int_1^\infty xe^{-x^2}dx$ converges or diverges. Evaluate it if it is convergent.

(b) Use the integral test, and your answer from (a), determine whether $\sum_{n=1}^{\infty} ne^{-n^2}$ converges or diverges. You must explicitly verify that the integral test applies to this series. No credit will be given if another test is used.

2.a. $$\int_1^\infty x e^{-x^2} dx = \lim_{n\to\infty} \int_1^n x e^{-x^2} dx = \lim_{n\to\infty} \left(-\frac{1}{2} e^{-x^2}\right]_1^n$$

$$= \lim_{n\to\infty} -\frac{1}{2} e^{-n^2} + \frac{1}{2} e^{-1} = \frac{1}{2} e^{-1}$$

b. Since the integral converges, the series converges

3. Determine whether the series $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{2n+3}$ is absolutely convergent, conditionally convergent or divergent. You must clearly explain your reasoning.

It's conditionally convergent.

Convergent : A.S.T. $b_n = \frac{\sqrt{n}}{2n+3}$, b_n's decreasing

$$\lim_{n\to\infty} \frac{\sqrt{n}}{2n+3} = \lim_{n\to\infty} \frac{1}{2\sqrt{n}+\frac{3}{\sqrt{n}}} = 0$$

not absolutely convergent : Apply the L.C.T to $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n+3}$. Compare to the divergent p-series, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{2n+3}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}$$

4. Find the radius of convergence and the interval of convergence of the following series.

(a) $\sum_{n=1}^{\infty} n!(2x-1)^n$

(b) $\sum_{n=1}^{\infty} \frac{(x-a)^n}{nb^n}$, where a and b are positive constants.

a. Ratio test $\lim_{n\to\infty} \left| \frac{(n+1)!\,(2x-1)^{n+1}}{n!\;(2x-1)^n} \right| = \lim_{n\to\infty} n(2x-1) = \infty$

$\boxed{\text{I.o.c} \quad x = \frac{1}{2}}$

b. Ratio Test $\lim_{n\to\infty} \left| \frac{\frac{(x-a)^{n+1}}{(n+1)\,b^{n+1}}}{\frac{(x-a)^n}{n\,b^n}} \right| = \lim_{n\to\infty} \left| \frac{(x-a)^{n+1}}{(n+1)\,b^{n+1}} \cdot \frac{n\,b^n}{(x-a)^n} \right|$

$= \lim_{n\to\infty} \frac{|x-a|}{b} \cdot \frac{n}{n+1} = \frac{|x-a|}{b} < 1$. So $-b < x-a < b$ or

$a-b < x < a+b$. If $x = a+b$, $\sum \frac{1}{n}$ divergent. If $x = a-b$, then $\sum \frac{(-1)^n}{n}$ convergent. I.o.c $[a-b, a+b)$

$n=1$

5. Consider $x = t^2 + 1, y = e^{2t} - 1$.

 (a) Find $\frac{dy}{dx}$.

 (b) Determine the location of any horizontal tangents. If none exist, explain why.

 (c) Find $\frac{d^2y}{dx^2}$.

 (d) Determine ~~the values of t for which the curve is concave up.~~ the concavity when $t=10$

a. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2e^{2t}}{2t} = t^{-1}e^{2t}$

b. Since $2e^{2t} \neq 0$, there are no horizontal tangents

c. $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{-t^{-2}e^{2t} + 2t^{-1}e^{2t}}{2t} \cdot \frac{t^2}{t^2} = \frac{-e^{2t} + 2te^{2t}}{2t^3}$

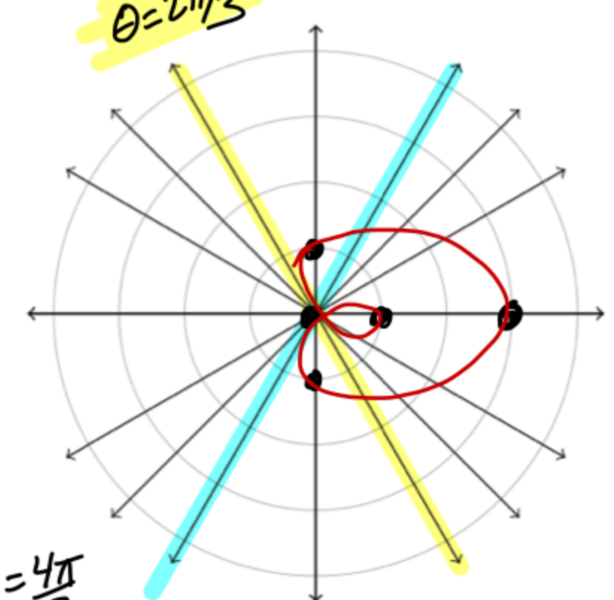
$$= \frac{e^{2t}(2t-1)}{2t^3}$$

$\left.\frac{d^2y}{dx^2}\right|_{t=10} = \frac{e^{20}(9)}{2\cdot 10^3} > 0$. So the graph is ccup

6. Consider the curve $r = 1 + 2\cos\theta$.

 (a) Sketch the curve $r = 1 + 2\cos\theta$. Include the coordinates of all x- and y-intercepts.

$\theta = 2\pi/3$



$\theta = \frac{4\pi}{3}$

θ	0	$\frac{\pi}{2}$	π	$3\pi/2$
r	3	1	-1	0

$0 = 1 + 2\cos\theta$

$\cos\theta = -\frac{1}{2}$, $\theta = \frac{2\pi}{3}, +\frac{4\pi}{3}$

 (b) Find the area enclosed by the inner loop.

$$A = \frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi} (1+2\cos\theta)^2\, d\theta$$

7. For each problem below, set up an integral(s) to find the quantity.

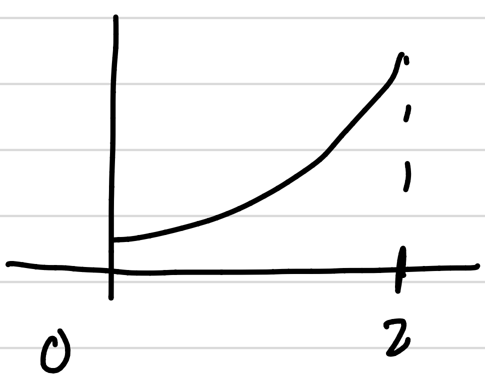
   (a) Find the mass of a wire that is 2 meters long (starting at $x = 0$) and has density $\rho(x) = 3x + 1$ grams per meter.

   (b) Let $\mathcal{R}$ be the region bounded by $y = e^x$ and $y = 0$, $0 \le x \le 2$. If the density of the region is given by $\rho = 5$, find the center of mass of R (or, equivalently, find the centroid of R.)

   (c) Recall that in the metric system force, F, is often measured in newtons (N) and work, W, is often measured in joules (j) or newton-meters ($N \cdot m$). Suppose a spring has a natural length of 15 cm and exerts a force of 8 N when stretched to a length of 20 cm. How much work is done stretching the spring from 15 cm to 25 cm?

a. $$m = \int_0^2 (3x+1)\,dx = \left.\frac{3x^2}{2} + x\right]_0^2 = 6+2 = 8\text{ g}$$

b.



$$m = 5\int_0^2 e^x\,dx$$

$$M_y = 5\int_0^2 x e^x\,dx$$

$$M_x = 5\int_0^2 \frac{(e^x)^2}{2}\,dx$$

$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}$$

c. Hooke's Law $F = kx$. So $8 = k \cdot 5$ or $k = \frac{8}{5}$.

$$W = \int_0^{10} \frac{8}{5}x\,dx$$