Midterm II Practice Problems

1. a. - use limit notation, correctly
   - use integral notation correctly

   b. $\int_2^5 \frac{3x}{x^2-4}\,dx = \lim_{a\to 2^-} \int_a^5 \frac{3x\,dx}{x^2-4} = \lim_{a\to 2^-} \left(\frac{3}{2}\ln(x^2-4)\Big]_a^5 \right)$

   $= \lim_{a\to 2^-} \left(\frac{3}{2}\ln(21) - \frac{3}{2}\underbrace{\ln(a-4)}_{\to -\infty} \right) = \infty$; diverges.

2. (a) $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}$

   (b) $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$, So the sequence converges.

   (c) $\sum_{n=1}^{\infty} \frac{n}{2n+1}$

   (d) The series diverges by the Divergence Test.

   - $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0.$ ← application of test

   (e) $S_1 = \frac{1}{3}$, $S_2 = \frac{1}{3} + \frac{2}{5} = \frac{11}{15}$, $S_3 = \frac{1}{3} + \frac{2}{5} + \frac{3}{7} = \frac{122}{105}$

   The sequence $S_1, S_2, S_3, S_4, \ldots$ diverges because the series $\sum \frac{n}{2n+1}$ diverges.

3. a. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3\sqrt{n}+\pi}$ converges by Alternating Series Test

Application of Test

- $b_n = \dfrac{1}{3\sqrt{n}+\pi}$; • $b_{n+1} = \dfrac{1}{3\sqrt{n+1}+\pi} < \dfrac{1}{3\sqrt{n+1}}$ $+ b_n$ decreasing

- $\displaystyle\lim_{n\to\infty} \frac{1}{3\sqrt{n}+\pi} = 0$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^e}$ converges by showing it's absolutely convergent

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^e}\right| = \sum_{n=1}^{\infty} \frac{1}{n^e}$$, a convergent p-series where $p = e > 1$.

c. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n+\ln(n)}$, diverges by limit comparison test

Compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, a divergent p-series.

$$\lim_{n\to\infty} \frac{\frac{5}{n+\ln(n)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{5n}{n+\ln(n)} = \lim_{n\to\infty} \frac{5}{1+\underbrace{\frac{\ln(n)}{n}}_{0}} = \underbrace{5}_{\text{a number} \neq 0}$$

d. $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(2n)!}$ Converges by the Ratio Test

$$\lim_{n\to\infty} \frac{\frac{10^{n+1}}{(2n+2)!}}{\frac{10^n}{(2n)!}} = \lim_{n\to\infty} \frac{10^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{10^n} = \lim_{n\to\infty} \frac{10}{(2n+2)(2n+1)} = 0 < 1$$

e. $\displaystyle\sum_{n=0}^{\infty} \frac{n2^n}{5^n}$ converges by the root test

$$\lim_{n\to\infty} \sqrt[n]{\frac{n2^n}{5^n}} = \lim_{n\to\infty} \frac{\overset{\nearrow 1}{\sqrt[n]{n}} \cdot 2}{5} = \frac{2}{5} < 1$$

f. $\displaystyle\sum_{n=2}^{\infty} \frac{\sin^3(n)}{n^2+1}$ converges $\boxed{\text{absolutely}}$ by (direct) comparison test

Compare to $\sum \frac{1}{n^2}$, a convergent p-series.

$|\sin^3(n)| \leq 1$ and $\frac{1}{n^2+1} \leq \frac{1}{n^2}$.

So $\left|\frac{\sin^3(n)}{n^2+1}\right| < \frac{1}{n^2}$. So $\sum \left|\frac{\sin^3(n)}{n^2+1}\right|$ converges.

(4) Apply Integral Test to $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1}$

$$\int_0^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_0^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \arctan x \Big]_0^b$$

$$= \lim_{b\to\infty} \arctan b - \arctan 0 = \pi/4.$$ So the series converges

(5) $$\sum_{n=2}^{\infty} \frac{(-2)^n}{7^n} = \sum_{n=2}^{\infty} \left(\frac{4}{49}\right)\left(\frac{-2}{7}\right)^{n-2} = \sum_{k=0}^{\infty} ar^k$$

The series is geometric with $a=\frac{4}{49}$ and $r=\frac{-2}{7}$.

It converges because $|r| = \left|\frac{-2}{7}\right| < 1$.

It converges to $$\frac{2/49}{1-(\frac{-2}{7})} = \frac{2/49}{\frac{9}{7}} = \frac{2}{49}\cdot\frac{7}{9} = \frac{2}{63}$$

(6) $$\sum_{n=0}^{\infty} \frac{1}{3^n}(x-1)^n = \sum_{n=0}^{\infty}\left(\frac{x-1}{3}\right)^n$$ ← geometric

So the series converges if and only if

$\left|\frac{x-1}{3}\right| < 1$. So $|x-1| < 3$.

$$-3 < x-1 < 3$$
$$-2 < x < 4$$

answer: $(-2,4)$

(7) $$\frac{2x}{3+x} = \frac{\frac{2}{3}x}{1+\frac{x}{3}} = \frac{\frac{2}{3}x}{1-(\frac{-x}{3})} = \sum_{n=0}^{\infty} \frac{2}{3}x\left(\frac{-x}{3}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2 \cdot x^{n+1}}{3^{n+1}}$$

8. $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

$f''(x) = 2^2 e^{2x}$

$f'''(x) = 2^3 e^{2x}$

$\vdots$

$f^{(n)}(x) = 2^n e^{2x}$

$f^{(n)}(1) = 2^n e^2$

Taylor Series

$$\sum_{n=0}^{\infty} \frac{2^n e^2}{n!}(x-1)^n$$