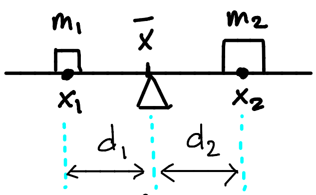
1. Intro to Moments and Center of Mass in One Dimension with Point Masses

Principle:
To achive balance



we need $|\bar{x}-x_1|\cdot m_1 = |\bar{x}-x_2|\cdot m_2$

Solve for $\bar{x}$ to get

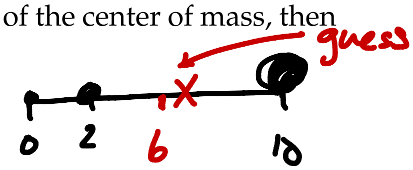
$$\bar{x} = \frac{m_1x_1+m_2x_2}{m_1+m_2}$$

$$\frac{\text{moment}}{\text{total mass}}$$

2. For the masses and locations below, (a) make a guess about the location of the center of mass, then (b) use the work from #1 above to find it precisely.

$m_1 = 2$ at $x_1 = 0$, $m_2 = 4$ at $x_2 = 2$, and $m_3 = 10$ at $x_3 = 10$.

(a)



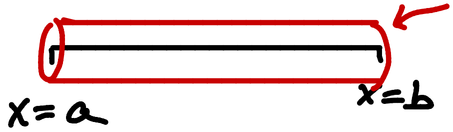
middle of 10 & 2

(b) moment $= M = \sum_{i=1}^{3} m_ix_i = 2\cdot 0 + 4\cdot 2 + 10\cdot 10 = 108$

mass $= m = \sum_{i=1}^{3} m_i = 2+4+10 = 16$

$$\bar{x} = \frac{M}{m} = \frac{108}{16} = 6.75$$

3. Intro to Moments and Center of Mass in One Dimension with Continuous Density



v. thin rod.
density $\rho(x)$.

§2.5: $m = \int_a^b \rho(x)\,dx$

$$\bar{x} = \frac{M}{m} = \frac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx}$$

moment $= M = \int_a^b \underbrace{x}_{x_i}\underbrace{\rho(x)\,dx}_{m_i}$

4. Compute the center of mass for a thin rod with density $\rho(x) = 12x^2$ kg/m assuming one end of the rod is at $x = 0$ m and the other is at $x = 2$ m.

Guess $\bar{x}$ way closer to $x=2$ than $x=0$.

$m = \int_0^2 12x^2\,dx = 4x^3\Big|_0^2 = 32$

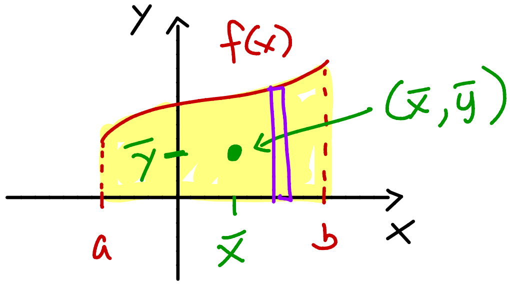
$\bar{x} = \frac{48}{32} = 1.5$

$M = \int_0^2 x(12x^2)\,dx = \int_0^2 12x^3\,dx$

$= 3x^4\Big]_0^2 = 48$

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

5. Intro to Moments and Center of Mass in Two Dimensions



$$\text{mass} = m = \int_a^b \underbrace{\rho}_{\text{mass/area}} \cdot \underbrace{f(x)\,dx}_{\text{area}}$$

$$\text{moment about } y = M_y = \int_a^b \underset{\text{location}}{x} \cdot \underbrace{\rho f(x)\,dx}_{\text{mass}} = \rho \int_a^b x f(x)\,dx$$

density ρ, units kg/m^2

Want $(\bar{x},\bar{y})$

$$\text{moment about } x = M_x = \int_a^b \underbrace{\tfrac{1}{2} f(x)}_{\text{location}} \underbrace{\rho f(x)\,dx}_{\text{mass}} = \frac{\rho}{2} \int_a^b (f(x))^2\,dx$$

6. Find the Center of Mass for the 2-dimensional regions below. Do you believe your answers?

(a) The region bounded by $y = \frac{1}{x}$, $y = 0$ $x = 1$, and $x = 5$. Assume $\rho = 2$. ← Important?

$$m = \int_1^5 2 \cdot \tfrac{1}{x} \cdot dx = 2\ln(x)\Big|_1^5 = 2\ln(5)$$

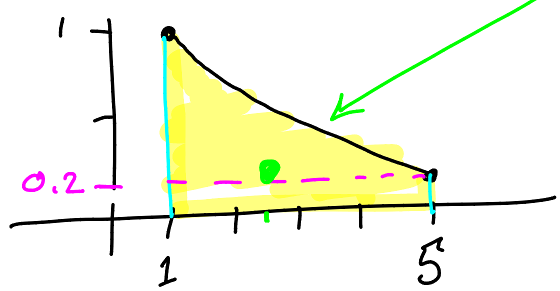
$$M_y = 2\int_1^5 x\left(\tfrac{1}{x}\right)dx = 2\int_1^5 dx = 8$$

$$M_x = 2 \cdot \tfrac{1}{2} \int_1^5 \left(\tfrac{1}{x}\right)^2 dx = \int_1^5 x^{-2}\,dx$$

$$-x^{-1}\Big|_1^5 = -\frac{1}{5} + \frac{1}{1} = \frac{4}{5}$$

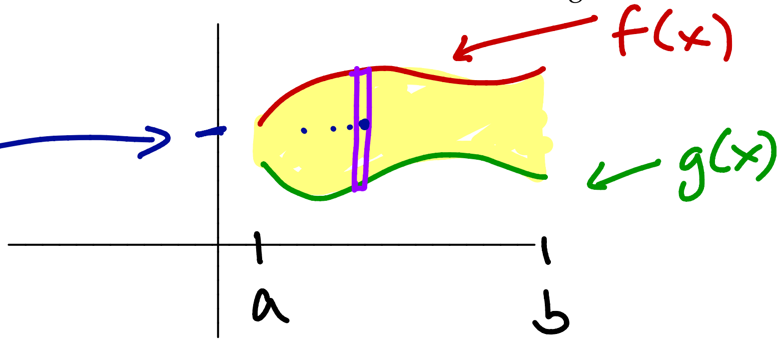
$$(\bar{x},\bar{y}) = \left(\frac{8}{2\ln(5)}, \frac{\frac{4}{5}}{2\ln(5)}\right)$$

$$\approx (2.48\ ,\ 0.249)$$



~~(b) The region bounded~~ by

7. Center of Mass in Two Dimensions Again



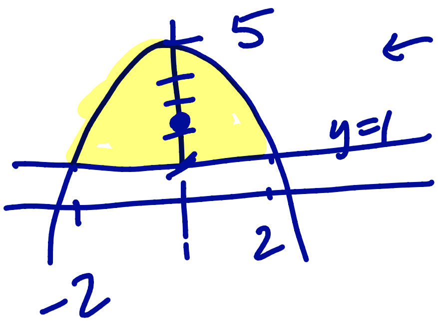
ρ - density

$$\text{mass} = m = \int_a^b \rho\,[f(x)-g(x)]\,dx$$

$$M_y = \int_a^b x\cdot(f(x)-g(x))\cdot\rho\,dx = \rho\int_a^b x\,(f(x)-g(x))\,dx$$

$$M_x = \int_a^b \underbrace{\left(\frac{f(x)+g(x)}{2}\right)}_{\text{location on y-axis}}\underbrace{(f(x)-g(x))\rho\,dx}_{\text{mass}} = \rho\int_a^b \left(f(x)^2 - g(x)^2\right)dx$$

8. Find the center of mass for the region bounded by $y = 5 - x^2$, $y = 1$. Assume ρ is constant. Sketch the region and see if your answer seems plausible.



$\leftarrow \bar{x} = 0$ should be! So $M_y = 0$! $\to$ Check

$$M_y = \rho\int_{-2}^{2} x(5-x^2-1)\,dx$$

$$= \rho\int_{-2}^{2}\underbrace{(4x-x^3)}_{\text{an odd fcn. so}}\,dx = 0$$

$$m = \int_{-2}^{2}\rho\,(5-x^2-1)\,dx = \rho\int_{-2}^{2}(4-x^2)\,dx$$

$$= 2\rho\int_0^2 (4-x^2)\,dx = 2\rho\left(4x-\tfrac{1}{3}x^3\right)\Big|_0^2 = 2\rho\left(8-\tfrac{8}{3}\right) = \frac{32\rho}{3}$$

$$M_x = \frac{\rho}{2}\int_{-2}^{2}\left((5-x^2)^2-(1)^2\right)dx = \frac{\rho}{2}\int_{-2}^{2}(24-10x^2+x^4)\,dx$$

$$= \rho\int_0^2 (24-10x^2+x^4)\,dx$$

$$= \rho \int_0^2 (24 - 10x^2 + x^4)\,dx$$

$$= \rho \left(24x - \frac{10}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^2$$

$$= \rho \left(48 - \frac{80}{3} + \frac{32}{5}\right) = \rho \, \frac{720 - 400 + 96}{15} = \rho \, \frac{416}{15}$$

So $$(\bar{x}, \bar{y}) = \left(0, \frac{\frac{416\rho}{15}}{\frac{32\rho}{3}}\right) = \left(0, \frac{416}{15} \cdot \frac{3}{32}\right)$$

$$= \left(0, \frac{13}{5}\right) = (0, 2.6)$$