Section 3.1: Integration by Parts

Nutshell

$\int u\,dv = u\cdot v - \int v\cdot du$

1. The Integration by Parts Formula

$f(x) = u(x)\cdot v(x)$

$f'(x) = u(x)\cdot v'(x) + u'(x)v(x).$

$u(x)\cdot v(x) = f(x) = \int f'(x)\,dx = \int u(x)\cdot v'(x)\,dx + \int u'(x)v(x)\,dx$

$u(x)v(x) - \int v(x)\cdot u'(x)\,dx = \int u(x)v'(x)\,dx$

2. Evaluate the integrals. What strategy is demonstrated?

(a) $\int xe^x\,dx$

$\begin{cases} u = x & dv = e^x dx \\ du = dx & v = e^x \end{cases}$

$= xe^x - \int e^x dx$

$= xe^x - e^x + C = (x-1)e^x + C$

Check: $y = (x-1)e^x$, $y' = 1\cdot e^x + (x-1)e^x = e^x + xe^x - e^x = xe^x$ ✓

Lesson: Make "u" term disappear.

(b) $\int \ln(x)\,dx$

$\begin{cases} u = \ln(x) & dv = dx \\ du = \frac{1}{x}dx & v = x \end{cases}$

$= x\ln(x) - \int x\cdot\frac{1}{x}dx$

$= x\ln(x) - \int dx = x\ln(x) - x + C.$

Lesson: Choose u to be everything but dx.

(c) $\int x^2\cos(x)\,dx$

$\begin{cases} u = x^2 & dv = \cos(x)\,dx \\ du = 2x\,dx & v = \sin(x) \end{cases}$

$= x^2\sin(x) - 2\int x\sin(x)\,dx$

$\begin{cases} u = x & dv = \sin(x)dx \\ du = dx & v = -\cos(x) \end{cases}$

$= x^2\sin(x) - 2\left[-x\cos(x) + \int\cos(x)\,dx\right]$

$= x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C$

Lesson: IBP more than 1 time.

Lesson: Solve for the integral.

$u = e^x \qquad dv = \cos(x)\,dx$

$du = e^x dx \qquad v = \sin(x)$

(d) $\int e^x \cos(x)\, dx = e^x \sin(x) - \int e^x \sin(x) dx$

$\begin{cases} u = e^x \\ du = e^x dx \end{cases} \qquad dv = \sin(x)\,dx \qquad v = -\cos(x)$

$$= e^x \sin(x) - \left[-e^x \cos(x) + \int e^x \cos(x)\, dx \right]$$

$$= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x) dx$$

solve for integral

$$2\int e^x \cos(x) dx = e^x \sin(x) + e^x \cos(x) = e^x(\sin(x) + \cos(x))$$

$$\int e^x \cos(x) dx = \frac{1}{2} e^x (\sin(x) + \cos(x)) + C$$

Lesson: IBP + definite integrals

(e) $\int_0^1 x e^{-3x}\, dx$

$\begin{cases} u = x \\ du = dx \end{cases} \qquad dv = e^{-3x} dx \qquad v = -\frac{1}{3} e^{-3x}$

$$= -\frac{1}{3} x e^{-3x} \Big]_0^1 + \frac{1}{3}\int_0^1 e^{-3x} dx$$

$$= -\frac{1}{3} e^{-3} + \frac{1}{3}\left(-\frac{1}{3} e^{-3x}\right)\Big|_0^1 = -\frac{1}{3} e^{-3} - \frac{1}{9}\left(e^{-3} - e^0\right)$$

$$= \frac{-4}{9} e^{-3} + \frac{1}{9} = \frac{1}{9}\left(1 - \frac{4}{e^3}\right)$$

(f) Find the area bounded between $f(x) = \arctan(x)$ and the y-axis between $x = 0$ and $x = 2$.

Lesson: IBP + u-substitution

$$A = \int_0^2 \arctan(x)\, dx$$

$u = \arctan(x) \qquad dv = dx$

$du = \frac{1}{1+x^2} dx \qquad v = x$

$$= x \arctan(x)\Big]_0^2 - \int_0^2 \frac{x\, dx}{1+x^2} = 2\arctan(2) - \frac{1}{2}\int_1^5 \frac{dw}{w}$$

$w = 1 + x^2 \qquad x = 0, w = 1$

$dw = 2x\,dx \qquad x = 2, w = 5$

$\frac{1}{2} dw = x\, dx$

$$= 2\arctan(x) - \frac{1}{2}\ln(w)\Big|_1^5 = 2\arctan(x) - \frac{1}{2}(\ln(5) - \ln(1))$$

$$= 2\arctan(x) - \frac{1}{2}\ln(5)$$