1. Compare the following three integrals:

u-sub

(a) $\int x\sqrt{9-x^2}\,dx = -\frac{1}{2}\int u^{\frac{1}{2}}\,du = -\frac{1}{2}\cdot\frac{2}{3}\,u^{\frac{3}{2}} = -\frac{1}{3}\,(9-x^2)^{3/2} + C$

let $u = 9-x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$\boxed{* \sin(2\theta) = 2\sin\theta\cos\theta}$

arcsine

(b) $\int \frac{dx}{\sqrt{9-x^2}} = \frac{1}{3}\int \frac{dx}{\sqrt{1-\left(\frac{x}{3}\right)^2}} = \frac{1}{3}\cdot 3\int\frac{du}{\sqrt{1-u^2}} = \arcsin(u) + C$

$= \arcsin\left(\frac{x}{3}\right) + C$

$u = \frac{x}{3}$

$du = \frac{1}{3}dx \Rightarrow 3\,du = dx$

(c) $\int \sqrt{9-x^2}\,dx = \int 3\cos\theta\cdot 3\cos\theta\,d\theta = 9\int\cos^2\theta\,d\theta = \frac{9}{2}\int(1+\cos(2\theta))\,d\theta$

$= \frac{9}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C \overset{*}{=} \frac{9}{2}\left[\theta + \frac{1}{2}(2\sin\theta\cos\theta)\right] + C$

trig sub

Let $x = 3\sin\theta$

$x^2 = 9\sin^2\theta$

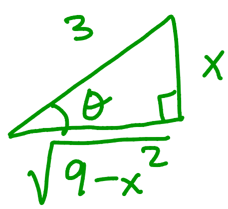
$9-x^2 = 9-9\sin^2\theta = 9\cos^2\theta$

$\sqrt{9-x^2} = \sqrt{9\cos^2\theta} = 3\cos\theta$

wowee!

$dx = 3\cos\theta\,d\theta$

$\boxed{\frac{x}{3} = \sin\theta,\ \theta = \arcsin\left(\frac{x}{3}\right);}$



$= \frac{9}{2}\left(\arcsin\left(\frac{x}{3}\right) + \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C$

$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C$

2. Summary: If $\sqrt{a^2-x^2}$ appears in an integrand (**and** other techniques do not work), then

try $x = a\sin\theta$

3. Evaluate $\int \frac{dx}{x^2\sqrt{4-x^2}} = \int\frac{2\cos\theta\,d\theta}{4\sin^2\theta\cdot 2\cos\theta} = \frac{1}{4}\int\csc^2\theta\,d\theta = -\frac{1}{4}\cot(\theta) + C$

$x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$

$4-x^2 = 4-4\sin^2\theta = 4\cos^2\theta$

$\sqrt{4-x^2} = 2\cos\theta$

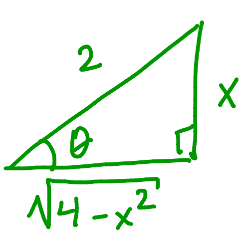
$\frac{x}{2} = \sin\theta$

$= -\frac{1}{4}\cdot\frac{\sqrt{4-x^2}}{x} + C$

$= -\frac{\sqrt{4-x^2}}{4x} + C$

$\cot(\theta) = \frac{\text{adj}}{\text{opp}}$

$= \frac{\sqrt{4-x^2}}{x}$



4. Compare the following integrals:

(a) $\int x\sqrt{9+x^2}\,dx$ $= \frac{1}{2}\int u^{1/2}\,du = \frac{1}{3}u^{3/2}+C = \frac{1}{3}(9+x^2)^{3/2}+C$

$u=9+x^2,\ du=2x\,dx$

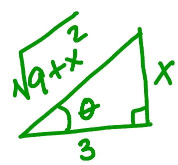
(b) $\int \frac{dx}{9+x^2}$ $= \frac{1}{9}\int \frac{dx}{1+(\frac{x}{3})^2} = \frac{1}{3}\arctan\left(\frac{x}{3}\right)+C$

(c) $\int \frac{dx}{\sqrt{9+x^2}}$ $= \int \frac{3\sec^2\theta\,d\theta}{3\sec\theta} = \int \sec\theta\,d\theta = \ln\left|\sec\theta+\tan\theta\right|+C$

Let $x=3\tan\theta$, $dx=3\sec^2\theta\,d\theta$

$\sqrt{9+x^2}=\sqrt{9+9\tan^2\theta}=3\sec\theta$

$\frac{x}{3}=\tan\theta$



$=\ln\left|\frac{\sqrt{9+x^2}}{3}+\frac{x}{3}\right|+C$

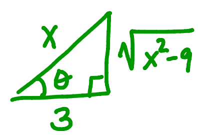
(d) $\int \frac{dx}{\sqrt{x^2-9}}$ $= \int \frac{3\sec\theta\tan\theta\,d\theta}{3\tan\theta} = \int \sec\theta\,d\theta = \ln\left|\sec\theta+\tan\theta\right|+C$

let $x=3\sec\theta$,

$dx=3\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-9}=3\tan\theta$

$\frac{x}{3}=\sec\theta$



$=\ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right|+C$

5. Summary:

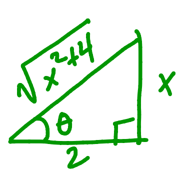
- If $\sqrt{a^2+x^2}$ appears in an integrand (**and** other techniques do not work), then

  $x=a\tan\theta$

- If $\sqrt{x^2-a^2}$ appears in an integrand (**and** other techniques do not work), then

  $x=a\sec\theta$

$\frac{x}{2}=\tan\theta$



6. Evaluate

(a) $\int \frac{dx}{(4+x^2)^2}$ $= \int \frac{2\sec^2\theta\,d\theta}{16\sec^4\theta} = \frac{1}{8}\int \cos^2\theta\,d\theta = \frac{1}{16}\int (1+\cos(2\theta))\,d\theta = \frac{1}{16}\left(\theta+\frac{1}{2}\sin(2\theta)\right)+C$

$x=2\tan\theta$

$dx=2\sec^2\theta\,d\theta$

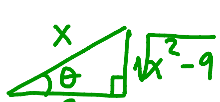
$4+x^2=4\sec^2\theta$

$= \frac{1}{16}\theta+\frac{1}{16}\sin\theta\cos\theta+C = \frac{1}{16}\arctan\left(\frac{x}{2}\right)+\frac{1}{16}\cdot\frac{x}{\sqrt{x^2+4}}\cdot\frac{2}{\sqrt{x^2+4}}$

$= \frac{1}{16}\left(\arctan\left(\frac{x}{2}\right)+\frac{2x}{x^2+4}\right)+C$

(b) $\int \frac{dx}{(x^2-9)^{3/2}}$

$x=3\sec\theta$



$= \int \frac{3\sec\theta\tan\theta\,d\theta}{27\tan^3\theta} = \frac{1}{9}\int \frac{\sec\theta\,d\theta}{\tan^2\theta} = \frac{1}{9}\int \frac{\cos\theta\,d\theta}{\sin^2\theta} = -\frac{1}{9}(\sin\theta)^{-1}+C$

$= -\frac{1}{9}\cdot\frac{x}{\sqrt{x^2-9}}$