1. (a motivating example) $\int \frac{3x}{x^2 - x - 2}\,dx$ $= \int \left(\frac{2}{x-2} + \frac{1}{x+1}\right)dx$

$\boxed{= 2\ln|x-2| + \ln|x+1| + C}$

$\boxed{\text{I Claim}}$

$$\frac{3x}{(x-2)(x+1)} = \frac{\overset{2}{A}}{x-2} + \frac{\overset{1}{B}}{x+1} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

So $3x = A(x+1)+B(x-2)$ Find A and B...

equate coefficients

$3x + 0 = (A+B)x + (A-2B)$

$3 = A+B$
$0 = A-2B$
2 eq, 2 unkn.

$3 = 3B$ or $B=1$

So $A=2$

strategic substitution

pick $x=-1$: $-3=-3B$, $B=1$

pick $x=2$: $6=3A$, $A=2$

2. Evaluate $\int \frac{7x+3}{x^3 - 4x^2 + 3x}\,dx$ $= \int\left(\frac{1}{x} + \frac{4}{x-3} - \frac{5}{x-1}\right)dx$

$= \boxed{\ln|x| + 4\ln|x-3| - 5\ln|x-1| + C}$

$$x^3 - 4x^2 + 3x = x(x^2 - 4x + 3) = x(x-3)(x-1)$$

$$\frac{7x+3}{x(x-3)(x-1)} = \frac{\overset{1}{A}}{x} + \frac{\overset{4}{B}}{x-3} + \frac{\overset{-5}{C}}{x-1}$$

So
$7x+3 = A(x-3)(x-1) + B(x)(x-1) + C(x)(x-3)$

Strategic Substitution

$x=0$: $3 = 3A$, $A=1$

$x=1$: $10 = -2C$, $C=-5$

$x=3$: $24 = 6B$, $B=4$

3. Summary of Partial Fraction Decomposition Structure

factor in denominator	linear $ax+b$	linear repeated $(ax+b)^k$	quadratic ax^2+bx+c
term in decomp	$\frac{A}{ax+b}$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_k}{(ax+b)^k}$	$\frac{Ax+B}{ax^2+bx+c}$

4. Rewrite the expression $\frac{x^3-2x^2+3x-6}{(x-1)^2}$ using partial fraction decomposition and use the decomposition to evaluate the integral $\int \frac{x^3-2x^2+3x-6}{(x-1)^2}\,dx$ =

$$\frac{x^3-2x^2+3x-6}{x^2-2x+1} = x + \frac{2x-6}{(x-1)^2}$$

$$\begin{array}{r} x \\ x^2-2x+1 \overline{\smash{)}\, x^3-2x^2+3x-6} \\ \underline{x^3-2x^2+x} \\ 2x-6 \end{array}$$

$$= \int\left(x + \frac{2}{x-1} + \frac{-4}{(x-1)^2}\right)dx$$

$$= \boxed{\frac{1}{2}x^2 + 2\ln|x-1| + 4\,(x-1)^{-1} + C}$$

$$\frac{2x-6}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$2x-6 = A(x-1)+B$$
$$= Ax + B - A$$

equate coeff

$2 = A$

$-6 = B-A = B-6$

$B = -4$

5. Rewrite the expression $\frac{4x^3+2x-1}{x^4+x^2}$ using partial fraction decomposition and use the decomposition to evaluate the integral $\int \frac{4x^3+2x-1}{x^4+x^2}\,dx$

$$\frac{4x^3+2x-1}{x^2(x^2+1)} = \frac{\overset{2}{A}}{x} + \frac{\overset{-1}{B}}{x^2} + \frac{\overset{2}{C}x+\overset{1}{D}}{x^2+1}$$

$$= \int\left(\frac{2}{x} - \frac{1}{x^2} + \frac{2x+1}{x^2+1}\right)dx = *$$

$$4x^3+2x-1 = A(x)(x^2+1) + \overset{-1}{B}(x^2+1) + (Cx+D)x^2$$

$$\boxed{\begin{array}{l} x=0, \\ -1 = B \end{array}}$$

$$= (A+C)x^3 + (B+D)x^2 + Ax + B$$

$$\overset{*}{=} 2\ln|x| + x^{-1} + \int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx$$

$A = 2$

$C = 2$

$B+D = 0$

So $D = 1$

$$\boxed{= 2\ln|x| + \frac{1}{x} + \ln|x^2+1| + \arctan(x) + C}$$