Section 3.7: Improper Integrals (Day 2)

Compute these integrals with friends! Please carefully write the limit, for example

$$\int_1^\infty \frac{dx}{x^2} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^2} = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty} 1 - \frac{1}{t} = 1$$

1. $\displaystyle\int_2^\infty \frac{1}{9+x^2}\,dx = \lim_{b\to\infty}\left(\frac{1}{9}\int_2^b \frac{dx}{1+(x/3)^2}\right) = \lim_{b\to\infty}\left(\frac{1}{3}\arctan\left(\frac{x}{3}\right)\Big|_2^b\right)$

as $b\to\infty$
$\arctan\left(\frac{b}{3}\right) \to \frac{\pi}{2}$

$$= \lim_{b\to\infty}\left(\frac{1}{3}\left(\arctan\frac{b}{3} - \arctan\frac{2}{3}\right)\right) = \frac{1}{3}\left(\frac{\pi}{2} - \arctan\left(\frac{2}{3}\right)\right)$$

Converges

2. $\displaystyle\int_{-\infty}^0 e^x\,dx = \lim_{a\to-\infty}\left(\int_a^0 e^x\,dx\right) = \lim_{a\to-\infty}\left(e^x\Big|_a^0\right) = \lim_{a\to-\infty}\left(e^0 - e^a\right) = \lim_{a\to-\infty}\left(1 - e^a\right)$

as $a\to-\infty$
$e^a \to 0$

$$= 1 - 0 = 1$$

Converges

3. $\displaystyle\int_0^1 \frac{1}{\sqrt[4]{x}}\,dx = \lim_{a\to0^+}\left(\int_a^1 x^{-1/4}\,dx\right) = \lim_{a\to0^+}\left(\frac{4}{3}x^{\frac{3}{4}}\Big|_a^1\right) = \lim_{a\to0^+}\left(\frac{4}{3}\left(1 - a^{\frac{3}{4}}\right)\right)$

$$= \frac{4}{3}(1-0) = \frac{4}{3}$$

Converges

4. $\displaystyle\int_0^1 \ln t\,dt = \lim_{a\to0^+}\left(\int_a^1 \ln t\,dt\right) = \lim_{a\to0^+}\left(t\ln t - t\Big|_a^1\right)$

aside: (IBP)

$\int \ln(t)\,dt = t\ln t - \int dt = \underline{\underline{t\ln t - t}}$

$u = \ln t \quad dv = dt$

$du = \frac{1}{t}\,dt \quad v = t$

$$= \lim_{a\to0^+}\left(\left(1\cdot\ln(1) - 1\right) - \left(a\ln(a) - a\right)\right)$$

$$= \lim_{a\to0^+}\left(-1 - \boxed{a\ln(a)} + a\right)$$

$$= -1 - 0 + 0 = -1$$

See attached sheet!

Converges

L'Hôpital's Rule to the rescue

$$\lim_{a\to 0^+} \boxed{a\ln a} = \lim_{a\to 0^+} \frac{\ln a}{a^{-1}} \overset{H}{=} \lim_{a\to 0^+} \frac{\frac{1}{a}}{-a^{-2}} = \lim_{a\to 0^+} \frac{1}{a}\cdot\frac{-a^2}{1}$$

$$= \lim_{a\to 0^+} -a = 0.$$

5. $\int_1^2 \frac{dx}{1-x} = \lim_{a\to 1^+}\left(\int_a^2 \frac{dx}{1-x}\right) = \lim_{a\to 1^+}\left(-\ln|1-x|\Big]_a^2\right) = \lim_{a\to 1^+}\left(\underbrace{-\ln|-1|}_{0} + \ln|1-a|\right)$

$= \lim_{a\to 1^+}\left(\ln|1-a|\right) = \lim_{a\to 1^+} \ln(a-1) = -\infty$ **divergES**

6. $\int_0^\infty e^x e^{-sx}\,dx = \lim_{b\to\infty}\left(\int_0^b e^x e^{-sx}\,dx\right) = \lim_{b\to\infty}\left(\int_0^b e^{(1-s)x}\,dx\right)$

$= \lim_{b\to\infty}\left(\frac{1}{1-s}\cdot e^{(1-s)x}\Big]_0^b\right) = \lim_{b\to\infty}\left(\frac{e^{(1-s)b}}{1-s} - \frac{1}{1-s}\right) = \begin{cases} \frac{1}{s-1} & \text{if } s<1 \\ \infty & \text{if } s>1 \end{cases}$

converges for some s-values and diverges for others.

7. $\int_0^\pi \tan x\,dx = \int_0^{\pi/2} \tan x\,dx + \int_{\pi/2}^{\pi} \tan x\,dx$

If ONE of these diverges the WHOLE diverges.
So, pick the first.

• $\int_0^{\pi/2} \frac{\sin x}{\cos x}\,dx = \lim_{b\to\frac{\pi}{2}^-}\left(\int_0^b \frac{\sin x}{\cos x}\,dx\right) = \lim_{b\to\frac{\pi}{2}^-}\left(-\ln|\cos x|\Big]_0^b\right) = \lim_{b\to\frac{\pi}{2}^-}\left(-\ln(\cos b) + \underbrace{\ln(\overset{1}{\cos 0})}_{0}\right)$

$= \lim_{b\to\frac{\pi}{2}^-}\left(-\ln|\cos b|\right) = \infty$ **diverges**

as $b\to \pi/2^-$, $\cos(b)\to 0^+$
so $\ln(\cos(b)) \to -\infty$

8. $\int_2^\infty \frac{dx}{x\ln^3 x} = \lim_{b\to\infty}\left(\int_2^b \frac{(\ln x)^{-3}\,dx}{x}\right) = \lim_{b\to\infty}\left(-\frac{1}{2}(\ln x)^{-2}\Big]_2^b\right)$

$= \lim_{b\to\infty}\left(-\frac{1}{2}\left(\frac{1}{(\ln b)^2} - \frac{1}{(\ln 2)^2}\right)\right) = -\frac{1}{2}\left(0 - \frac{1}{(\ln 2)^2}\right) = \frac{1}{2(\ln 2)^2}$

Converges