1. What is an improper integral and how to we handle them?

- $\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$
- $\int_{-\infty}^{b} f(x) = \lim_{a\to-\infty} \int_a^b f(x)$
- $\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$
- $\int_a^b g(x)dx = \lim_{t\to b^-} \int_a^t g(x)\,dx$

f(x) not defined at x=a
g(x) not defined at x=b

2. Evaluate the improper integrals below:

(a) $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln(x)\Big]_1^b = \lim_{b\to\infty} \left(\ln(b)-\ln(1)\right)$

$= \lim_{b\to\infty} \ln(b) = +\infty$. The integral diverges.

(b) $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left(\int_1^b x^{-2}\,dx\right) = \lim_{b\to\infty} \left(-x^{-1}\Big]_1^b\right) = \lim_{b\to\infty} \left[-b^{-1}-(-1^{-1})\right]$

$= \lim_{b\to\infty} -\frac{1}{b}+1 = 1$. So, the integral converges to 1.

* Interpret (a) and (b) geometrically.

3. Use the integrals above to decide if the integrals below converge or diverge. Write a complete sentence explaining your reasoning.

(a) $\displaystyle\int_1^{\infty} \frac{10}{\sqrt{x}}\,dx$. Thinking: on $[1,\infty]$, $\sqrt{x} \le x$. So $\frac{1}{\sqrt{x}} \ge \frac{1}{x}$

This integral diverges because $\frac{10}{\sqrt{x}} > \frac{1}{x}$ on $[1,\infty)$ and $\int_1^{\infty} \frac{1}{x}\,dx$ diverges.

(b) $\displaystyle\int_1^{\infty} \frac{1}{x^2+20x}\,dx$. Thinking $\frac{1}{x^2+20x} < \frac{1}{x^2}$ on $[1,\infty)$.

This integral converges because $\frac{1}{x^2+20x} < \frac{1}{x^2}$ on $[1,\infty)$ and $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges.

4. Evaluate the improper integrals below:

(a) $\displaystyle\int_3^9 \frac{dx}{(3-x)^2} = \lim_{t\to 3^+}\left(\int_t^9 (3-x)^{-2}\,dx\right) = \lim_{t\to 3^+}\left((3-x)^{-1}\Big]_t^9 \right)$

$$= \lim_{t\to 3^+}\left(\frac{1}{3-9} - \frac{1}{3-t}\right) = \lim_{t\to 3^+}\left(-\frac{1}{3} - \frac{1}{3-t}\right) = +\infty$$

as $t \to 3^+$

$3-t \to 0^-$

So $\frac{1}{3-t} \to -\infty$, So $-\left(\frac{1}{3-t}\right) \to +\infty$

So the integral diverges.

(b) $\displaystyle\int_0^6 \frac{1}{\sqrt{6-x}}\,dx = \lim_{t\to 6^-}\left(\int_0^t (6-x)^{-1/2}\,dx\right)$

$$= \lim_{t\to 6^-}\left(-2(6-x)^{1/2}\Big]_0^t\right) = \lim_{t\to 6^-}\left(-2\sqrt{6-t} + 2\sqrt{6}\right)$$

$$= -2\cdot 0 + 2\sqrt{6} = 2\sqrt{6}.$$

So the integral converges to $2\sqrt{6}$.