1. Things to Know by the end of this section:
   (a) what a sequence is
   (b) how to read and use sequence notation
   (c) what it means to "Find a formula for the nth term" and how to find it
   (d) what it means for a sequence to converge or diverge
   (e) how to use the many different techniques for determining if a sequence converges or diverges
   (f) what $n!$ means
   (g) what the following terms mean when referencing a sequence: bounded, monotone, increasing, decreasing

2. A infinite sequence is an ordered list of numbers.

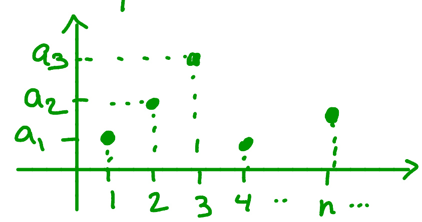
first term

$a_1, a_2, a_3, \dots, a_n, \dots$ ← the n^{th} term

second term

or $\{a_n\}_{n=1}^{\infty}$

picture $\{a_n\}$



3. For each sequence below, write the first 5 terms and graph them.

   (a) $\left\{\frac{n+2}{2n}\right\}_{n=1}^{\infty}$

n	1	2	3	4	5
a_n	$\frac{1+2}{2\cdot 1} = \frac{3}{2}$	$\frac{4}{4} = 1$	$\frac{5}{6}$	$\frac{6}{8}$	$\frac{7}{10}$

$$\boxed{\frac{3}{2}, 1, \frac{5}{6}, \frac{6}{8}, \frac{7}{10}, \dots}$$

   (b) $a_n = 3\left(\frac{-1}{2}\right)^{n-1}$ for $n \geq 1$

n	1	2	3	4	5
a_n	$3\left(-\frac{1}{2}\right)^0 = 3$	$3\left(-\frac{1}{2}\right) = -\frac{3}{2}$	$\frac{3}{4}$	$\frac{-3}{8}$	$\frac{3}{16}$

$$\boxed{3, -\frac{3}{2}, \frac{3}{4}, \frac{-3}{8}, \frac{3}{16}, \dots}$$

   (c) $a_1 = 5$ and $a_n = 2 \cdot a_{n-1} + 1$ ← recursive definition

n	1	2	3	4	5
a_n	5	$2\cdot a_1 + 1 = 2\cdot 5 + 1 = 11$	$2\cdot a_2 + 1 = 2\cdot 11 + 1 = 23$	$2\cdot a_3 + 1 = 2\cdot 23 + 1 = 47$	95

$$\boxed{5, 11, 23, 47, 95, \dots}$$

4. Find a formula for 3c.

strategy: explore. look for patterns.

$a_1 = 5$

$a_2 = 2\cdot 5 + 1$

$a_3 = 2(2\cdot 5 + 1) + 1$
$= 2^2\cdot 5 + 2 + 1$

$a_4 = 2(2^2\cdot 5 + 2 + 1) + 1$
$= 2^3\cdot 5 + 2^2 + 2 + 1$

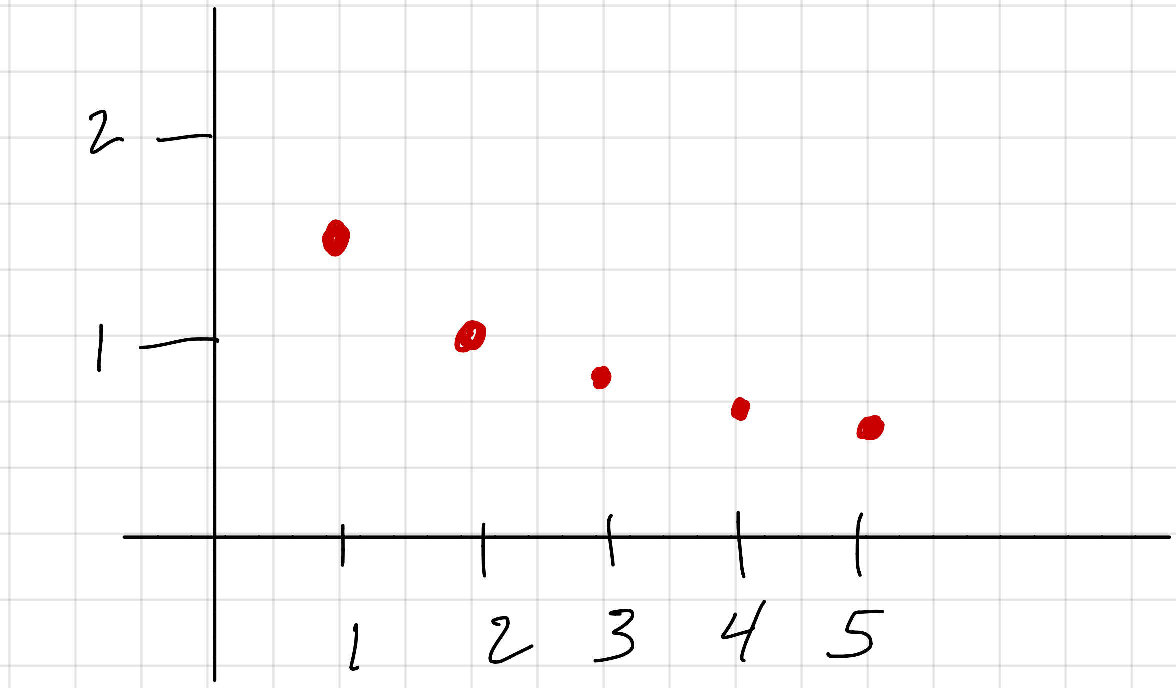
$a_5 = 2(2^3\cdot 5 + 2^2 + 2 + 1)$
$= 2^4\cdot 5 + 2^3 + 2^2 + 2 + 1$

$a_n = 2^{n-1}\cdot 5 + 2^{n-2} + 2^{n-3} + \dots + 2^2 + 2 + 1$
$= 2^{n-1}\cdot 5 + \sum_{k=0}^{n-2} 2^k$

We'll have nice ways of dealing with ~~this~~ this

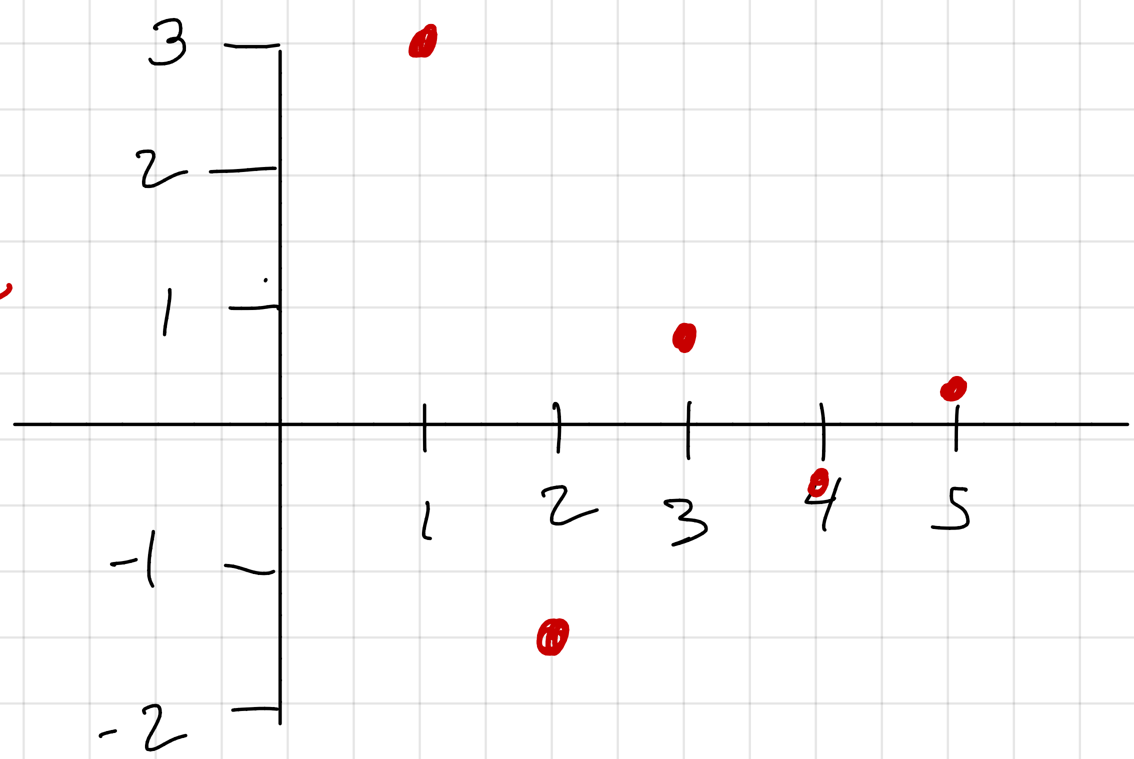
$\frac{3}{2}, 1, \frac{5}{6}, \frac{6}{8}, \frac{7}{10}, \ldots$

monotone
decreasing
bounded



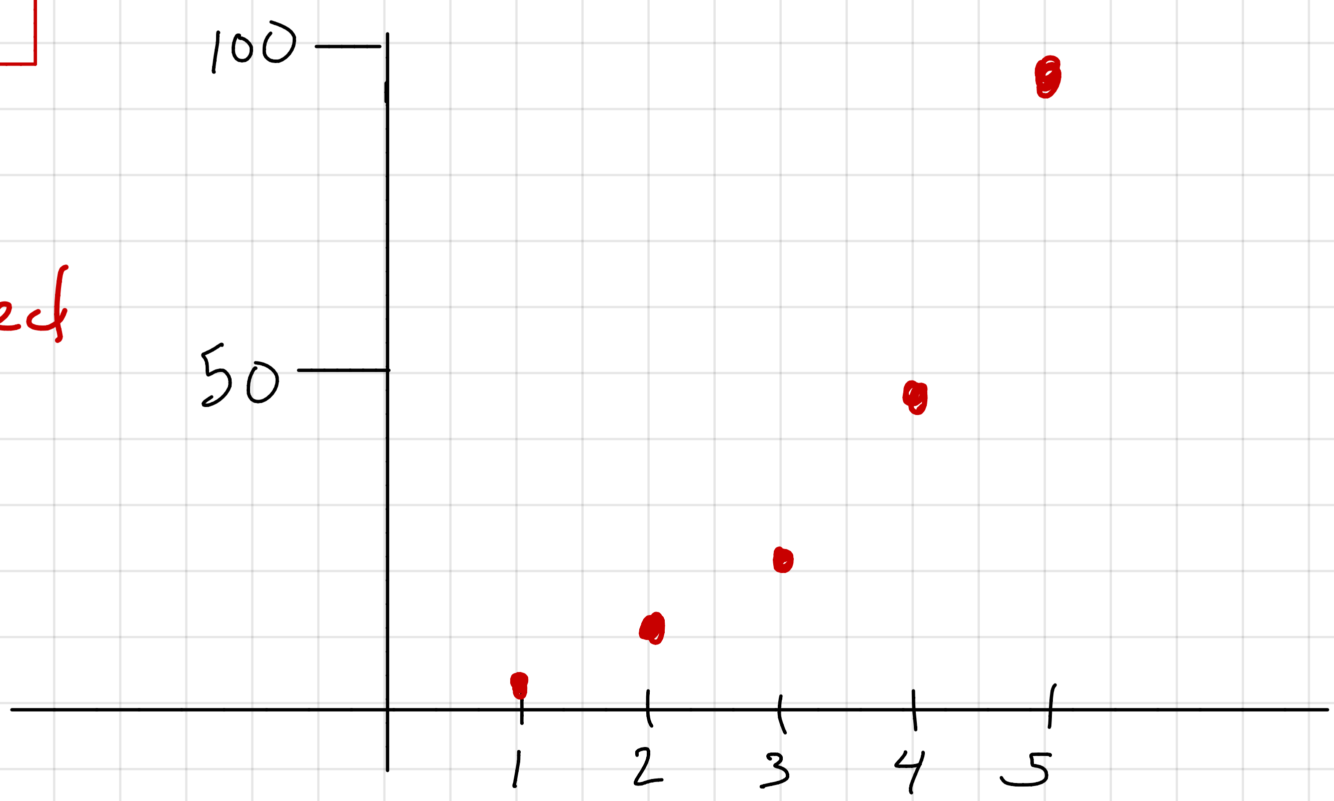
$3, -\frac{3}{2}, \frac{3}{4}, \frac{-3}{8}, \frac{3}{16}, \ldots$

not monotone
bounded



$5, 11, 23, 47, 95, \ldots$

monotone
increasing
not bounded



5. Definition: The symbol $n!$ (or "n factorial") means $\boxed{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n}$

So
$$1! = 1$$
$$2! = 1 \cdot 2 = 2$$
$$3! = 1 \cdot 2 \cdot 3 = 6$$
$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 3! \cdot 4 = 24$$

6. Find a formula for the sequence $a_1 = 1$ and $a_n = 3 \cdot a_{n-1}/n$

$$1 = a_1 = 1 = \frac{3^{1-1}}{1!}$$
$$\frac{3}{2} = a_2 = \frac{3 \cdot 1}{2} = \frac{3^{2-1}}{2!}$$
$$\frac{3}{2} = \frac{9}{6} = a_3 = 3\left(\frac{3}{2}\right) \cdot \frac{1}{3} = \frac{3^2}{2 \cdot 3} = \frac{3^{3-2}}{3!}$$
$$\frac{9}{8} = a_4 = 3\left(\frac{3^2}{2 \cdot 3}\right)\left(\frac{1}{4}\right) = \frac{3^3}{2 \cdot 3 \cdot 4} = \frac{3^{4-1}}{4!}$$

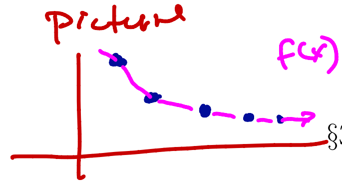
$$a_n = \frac{3^{n-1}}{n!}$$

7. sequences and convergence

- $\{a_n\}$ a sequence, we say $\boxed{\lim_{n \to \infty} a_n = L}$ if the terms of the sequence gets closer and closer to L as we go further out on sequence. In this case, we say $\{a_n\}$ converges to L, or $\{a_n\}$ is convergent
- If $\lim_{n \to \infty} a_n$ does not exist, we say $\{a_n\}$ is divergent
- Return to pictures for 3 a b c. Do you think these converge or diverge?
- How much do the first two terms matter when guessing whether these sequence converge? The first 10 terms? 100 terms?
- discrete vs. continuous functions
- Observation: Given sequence $\{a_n\}$, if we can find $f(x)$ (well behaved) such that $f(n) = a_n$ for all $n = 1, 2, 3, 4, \ldots$, then we can our Calc I techniques to find limits by:

$$\lim_{n \to \infty} a_n = \lim_{x \to \infty} f(x)$$

Picture



8. Find the limit of each of the following sequences or show that it diverges.

(a) $\left\{ \pi + \dfrac{100}{n} \right\}$

converges

$$\lim_{n\to\infty} \left(\pi + \underbrace{\frac{100}{n}}_{\to 0 \text{ by number sense}}\right) = \pi$$

(b) $\left\{ \dfrac{100n^2 + \sqrt{n}}{n - 3n^2} \right\}$

$$\lim_{n\to\infty} \frac{100n^2 + \sqrt{n}}{n - 3n^2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{100 + \frac{1}{n^{3/2}}}{\frac{1}{n} - 3} = \frac{-100}{3}$$

Converges

(c) $\left\{ \dfrac{e^n}{n^2} \right\}$

$$\lim_{n\to\infty} \underbrace{\frac{e^n}{n^2}}_{\frac{\infty}{\infty}} \overset{H}{=} \lim_{n\to\infty} \underbrace{\frac{e^n}{2n}}_{\frac{\infty}{\infty}} \overset{H}{=} \lim_{n\to\infty} \frac{e^n}{2} = \infty$$

diverges.

(d) $\left\{ \left(1 + \dfrac{1}{n}\right)^n \right\}$

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e^1 = e$$

$$\lim_{n\to\infty} n \ln\left(1 + \frac{1}{n}\right) = \lim_{n\to\infty} \underbrace{\frac{\ln(1+n^{-1})}{n^{-1}}}_{\frac{\infty}{\infty}} \overset{H}{=} \lim_{n\to\infty} \frac{\frac{-n^{-2}}{1+n^{-1}}}{-n^{-2}}$$

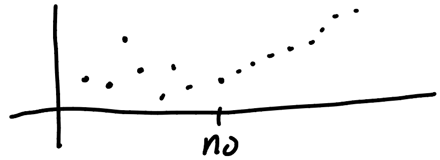
$$= \lim_{n\to\infty} \frac{1}{1+n^{-1}} = \boxed{1}$$

9. Definitions: The sequence $\{a_n\}$ is:

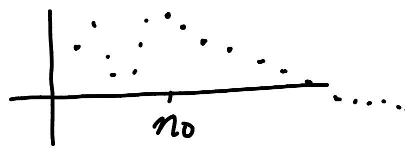
(a) bounded if there are numbers M_1 and M_2 so that

$$M_1 \le a_n \le M_2 \quad \text{for all } n = 1, 2, 3, \ldots$$

(b) increasing for $n \geq n_0$ if $a_n \le a_{n+1}$ for all integers $n \geq n_0$



(c) decreasing for $n \geq n_0$ if $a_n \geq a_{n+1}$ for all integers $n \geq n_0$



(d) monotone for $n \geq n_0$ if $\{a_n\}$ is either an increasing or a decreasing sequence for $n \geq n_0$

10. Monotone Convergence Theorem

If $\{a_n\}$ is bounded and there is some n_0 such that $\{a_n\}$ is monotone for all $n \geq n_0$, then $\{a_n\}$ converges.

11. Are the sequences below bounded? Monotone?

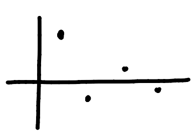
(a) $a_n = \frac{(-1)^{n-1}}{3^n}$ for $n \geq 1$

$a_1 = \frac{1}{3}$

$a_2 = -\frac{1}{9}$

$a_3 = \frac{1}{27}$

$a_4 = -\frac{1}{81}$



• bounded? Yes. $-\frac{1}{3} \le a_n \le \frac{1}{3}$

• monotone? No. It bounces back & forth between positive and negative values.

(b) $a_n = \frac{3^n}{n!}$ for $n \geq 1$

$a_1 = 3$

$a_2 = \frac{9}{2}$

$a_3 = \frac{27}{6} = \frac{9}{2}$

$a_4 = \frac{81}{24} = \frac{27}{8}$

$a_5 = \frac{243}{5!} = \frac{27}{40}$

think about a_n: $a_n = \frac{3^n}{n!} = \left(\underbrace{\frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{3}}_{\text{some \#}}\right)\left(\underbrace{\frac{3}{4} \cdot \cdots \cdot \frac{3}{n-1} \cdot \frac{3}{n}}_{\text{all } < 1}\right)$

So, for $n \geq 3$ $a_{n+1} = a_n \cdot \frac{3}{n+1} < a_n$ because $\frac{3}{n+1} < 1$. So $a_{n+1} < a_n$. So the sequence is decreasing for $n \geq 3$. Bounded? Yes. $0 \le a_n \le 3$.