Things to Know by the end of this section:

a. how to use sigma notation *with facility*.

b. what is meant by a *series*, especially as compared to a *sequence* (from §5.1).

c. what is meant by *a sequence of partial sums of a series* and how to find it.

d. what it means to say a series converges.

e. what a *geometric series* is and how to determine whether or not it converges.

f. what a *telescoping series* is and how to determine whether or not it converges.

g. develop facility with the terminology of this section including all the terms above and things like: the terms of the series, the $k$th term in the series and the $k$th partial sum of the series.

1. An infinite series is a sum of infinitely many terms.

$$\underbrace{\sum_{n=1}^{\infty} a_n}_{\text{series}} = \underbrace{a_1 + a_2 + a_3 + \dots + a_n + \dots}_{\text{series}}$$

$a_2$ is a term in the series

$a_n$ is the $n^{th}$ term in the series

2. The sequence of partial sums of a series is $\{S_n\}_{n=1}^{\infty}$ where

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_4 = a_1 + a_2 + a_3 + a_4$

$\vdots$

$S_k = a_1 + a_2 + a_3 + \dots + a_k = \sum_{n=1}^{k} a_k$

See the difference between the series and the $k^{th}$ partial sum?

3. For each series below, expand the sigma notation and then write the first 5 terms in its sequence of partial sums, $S_1, S_2, S_3, S_4, S_5$. (Use a calculating device to get a decimal or fraction representation of the partial sums.)

(a) $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots + \left(\frac{2}{3}\right)^n + \dots$

$S_1 = \frac{2}{3} = 0.\overline{6}$

$S_2 = \frac{2}{3} + \frac{4}{9} = 1.\overline{1}$

$S_3 = \frac{2}{3} + \frac{4}{9} + \frac{8}{27} = 1.\overline{407}$

$S_4 = \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} = 1.604938272$

$S_5 = \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{243} = 1.736625514$

$\{S_n\} = 0.\overline{6}, 1.\overline{1}, 1.\overline{407}, 1.604\dots, 1.736\dots, \dots$

(b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \dots = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots + \frac{1}{n(n+1)} + \dots$

$S_1 = \frac{1}{2} = 0.5$

$S_2 = \frac{1}{2} + \frac{1}{6} = 0.\overline{6} = 2/3$

$S_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = 0.75 = 3/4$

$S_4 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = 0.8 = 4/5$

$S_5 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = 0.83 = 5/6$

$\{S_n\} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots$

$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ partial frac

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{5}$ $= -\frac{1}{5} + \frac{1}{5} - \frac{1}{5} + \frac{1}{5} - \frac{1}{5} + \cdots$

$S_1 = -\frac{1}{5}$ $\{S_n\} = -\frac{1}{5}, 0, -\frac{1}{5}, 0, -\frac{1}{5}, 0, \ldots$

$S_2 = 0$

$S_3 = -\frac{1}{5}$

$S_4 = 0$

$S_5 = -\frac{1}{5}$

$\lim_{n\to\infty} S_n$ Does not exist

b/c it bounces back and forth between $-1/5$ and 0.

(d) $\sum_{n=1}^{\infty} \frac{n}{n^2+2}$ $= \frac{1}{3} + \frac{2}{4+2} + \frac{3}{9+2} + \frac{4}{16+2} + \frac{5}{25+2} + \cdots = \frac{1}{3} + \frac{2}{6} + \frac{3}{11} + \frac{4}{18} + \frac{5}{27} + \cdots$

$S_1 = \frac{1}{3} = 0.\overline{3}$ $\{S_n\} = \frac{1}{3}, \frac{2}{3}, 0.\overline{93}, 1.\overline{16}, 1.3468\ldots, \ldots$

$S_2 = \frac{1}{3} + \frac{2}{6} = \frac{2}{3} = 0.\overline{6}$

$S_3 = \frac{1}{3} + \frac{2}{6} + \frac{3}{11} = 0.\overline{93}$

$S_4 = \frac{1}{3} + \frac{2}{6} + \frac{3}{11} + \frac{4}{18} = 1.\overline{16}$

$S_5 = \frac{1}{3} + \frac{2}{6} + \frac{3}{11} + \frac{4}{18} + \frac{5}{25} = 1.3468\ldots$

$S_6 = 1.5046\ldots$

$S_7 = 1.6419\ldots$

4. **Definition:** Given the series $\sum_{n=1}^{\infty} a_n$, its sequence of partial sums is $\{S_n\}_{n=1}^{\infty}$.

- The series converges if $S_n$ converges. $\left(\lim_{n\to\infty} S_n = L\right)$
- The series diverges if $S_n$ diverges. $\left(\lim_{n\to\infty} S_n \text{ does not exist.}\right)$

5. Revisit the series in # 3 and determine (if possible!) whether the series converges or diverges. Show your work!