NOTE: The symbol **!!!** indicates that this series is one of the top three series to understand. These series will be used repeatedly in this and other classes.

1. **(!!!)** A geometric series has form $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \dots + ar^k + \dots$

- If $|r|<1$, then $\sum_{n=1}^{\infty} ar^{n-1}$ converges to $\frac{a}{1-r}$.

If $|r| \geq 1$, then the series $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

Why? See last page $\longrightarrow$

2. **Ex 1:** $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1}$

- geometric
- $a=1$
- $r=\frac{2}{3}$, $|r|<1$

Conclusion: $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} = \frac{1}{1-2/3} = 3$

Series converges

3. **Ex 2:** $\sum_{n=1}^{\infty} \frac{4^{n-1}}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3}\left[\frac{4}{3}\right]^{n-1}$.

- geometric
- $a=\frac{1}{3}$, $r=4/3$
- $|r| \geq 1$

So series diverges

4. A telescoping series is one for which most terms in S_k cancel leaving only a few terms at the beginning.

5. **Ex 3:** $\sum_{n=1}^{\infty} \frac{1}{(n+1)n} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{k} - \frac{1}{k+1}\right) + \dots$

Partial fractions:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$S_k = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \dots + \left(\cancel{\tfrac{1}{k-1}} - \cancel{\tfrac{1}{k}}\right) + \left(\cancel{\tfrac{1}{k}} - \frac{1}{k+1}\right)$$

$$= 1 - \frac{1}{k+1}$$

Now: $\lim_{k\to\infty} S_k = \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right) = 1$.

Conclusion: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$, converges.

7. (!!!) harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots+\frac{1}{k}+\dots$

It DIVERGES even though its terms get really small.

They do not get small fast enough. Why? $\longrightarrow$

8. For each series below, determine whether the series converges or diverges. If it converges, determine its sum. State the technique you are using.

(a) $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \sum_{n=1}^{\infty} \frac{2}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{2/3}{1-2/3} = \frac{2}{3}\cdot\frac{3}{1} = 2$

$a = 2/3$

$r = 2/3$

(b) $\sum_{n=1}^{\infty} 10\left(\frac{-3}{5}\right)^n = \sum_{n=1}^{\infty} 10\left(\frac{-3}{5}\right)\left(\frac{-3}{5}\right)^{n-1} = \frac{-6}{1-\left(\frac{-3}{5}\right)} = -6\cdot\frac{5}{8}$

$= -15/4$

$a = -6$

$r = -3/5$

(c) $\sum_{n=1}^{\infty}(e^{2/n} - e^{2/(n+1)}) = \underbrace{(e^2 - e^1)}_{n=1} + \underbrace{(e^1 - e^{2/3})}_{n=2} + \underbrace{(e^{2/3} - e^{2/4})}_{n=3} + \underbrace{(e^{2/4} - e^{2/5})}_{n=4} + \dots + \underbrace{(e^{2/k} - e^{2/k+1})}_{n=k} + \dots$

$S_k = (e^2 - e^1) + (e^1 - e^{2/3}) + (e^{2/3} - e^{2/4}) + \dots + (e^{2/k} - e^{2/k+1})$

$= e^2 - e^{2/k+1}$. $\lim_{k\to\infty} S_k = \lim_{k\to\infty} e^2 - e^{2/k+1} = e^2 - e^0 = \underline{e^2 - 1}$

So $\sum (e^{2/n} - e^{2/n+1}) = e^2 - 1$ converges

(d) $\sum_{n=1}^{\infty}\left[\left(\frac{2}{3}\right)^n + 10\left(\frac{-3}{5}\right)^n\right] = 2 - \frac{15}{4} = -7/4$

(e) $\sum_{n=1}^{\infty} \frac{\sin(\pi n/2)}{5} = \frac{1}{5} + 0 - \frac{1}{5} + 0 + \frac{1}{5} + 0 - \frac{1}{5} + \dots$

S_k bounces between $\frac{1}{5}$ and 0. So S_k diverges.

So series diverges

Given geometric series $\sum_{n=1}^{\infty} ar^{n-1}$,

its sequence of parial sums is:

$S_1 = a$

$S_2 = a + ar$

$S_3 = a + ar + ar^2$

$\vdots$

$S_k = a + ar + ar^2 + \dots + ar^{k-1}$

$\vdots$

Immediate Observation:

If $r=1$, $S_k = a+a+a+\dots+a = ka$.

If $r=-1$, $S_k = a-a+a-a+a \dots \pm a$.

Neither converges!

Observe: $rS_k = ar + ar^2 + ar^3 + \dots + ar^{k-1} + ar^k$

So $(1-r)S_k = S_k - rS_k = a + ar^k = a(1+r^k)$

Solve for $S_k = \dfrac{a(1+r^k)}{1-r}$.

With $r \neq \pm 1$, $\lim_{k\to\infty} \dfrac{a}{1-r}(1+r^k) = \begin{cases} \dfrac{a}{1-r} & \text{for } |r|<1 \\ \text{DNE} & \text{for } |r|>1 \end{cases}$

Why does $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge?

$$\sum_{n=1}^{\infty} \frac{1}{n} = \underbrace{1} + \underbrace{\frac{1}{2}} + \underbrace{\frac{1}{3} + \overset{\text{smallest term}}{\frac{1}{4}}}_{\text{2 terms}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \overset{\text{smallest term}}{\frac{1}{8}}}_{\text{4 terms}} + \underbrace{\frac{1}{9} + \dots + \overset{\text{smallest term}}{\frac{1}{16}}}_{\text{8 terms}} + \underbrace{\frac{1}{17} + \dots + \frac{1}{32}}_{\text{16 terms}} + \dots$$

$$\geq 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + 8\left(\frac{1}{16}\right) + 16\left(\frac{1}{32}\right) + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$