1. The comparison tests *depend* on knowledge of geometric series and p-series. For each, (i) give the form of the series, (ii) state the conditions under which it converges and diverges, (iii) give examples of convergent and divergent series of the given type.

(a) geometric series

(i) $\sum_{n=1}^{\infty} ar^{n-1}$ (ii) If $|r|<1$, $\sum ar^n$ converges
If $|r|\geq 1$, $\sum ar^n$ diverges

(iii) $\sum_{n=1}^{\infty} 100\left(\frac{2}{5}\right)^{n-1}$ converges, $\sum_{n=1}^{\infty}\left(\frac{8}{7}\right)^{n-1}$ diverges

(b) p-series

(i) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (ii) If $p>1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges
If $p\leq 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

(iii) $\sum_{n=1}^{\infty} \frac{1}{n^{10}}$ converges, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges

2. The Comparison Test

- For $n\geq N$, $0\leq a_n \leq b_n$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- For $n\geq N$ $0\leq b_n\leq a_n$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

[In practice, (1) You decide if you *think* $\sum a_n$ converges or diverges. (2) Then you go looking for a *suitable* $\sum b_n$]

3. Use the **comparison test** to determine whether the series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{3^n}{4^n+2^n}$

5 points to make

Answer: • Pick $b_n = \frac{3^n}{4^n}$.

• Since $4^n+2^n > 4^n$, $\frac{1}{4^n+2^n} < \frac{1}{4^n}$.

So $\frac{3^n}{4^n+2^n} < \frac{3^n}{4^n}$.

• Since $\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$ is a convergent geometric series, $\sum_{n=1}^{\infty}\frac{3^n}{4^n+2^n}$ converges by the Comparison Test.

work/thinking

- guess convergence
- need b_n so that $\sum b_n$ converges and $a_n \leq b_n$ (eventually)
- pick: $\sum b_n = \sum\left(\frac{3}{4}\right)^n$ convergent geo. series

(b) $\sum_{n=1}^{\infty} \frac{3}{5n-1}$

thinking

- Looks like p-series w/ $p=1$.
- guess divergent.
- check want $\frac{1}{n} < \frac{1}{5n-1}$

Nope! But $\frac{1}{5n} = \left(\frac{1}{5}\right)\left(\frac{1}{n}\right) < \frac{1}{5n-1}$

Answer: • Pick $b_n = \frac{1}{5n}$.

• Since $5n > 5n-1$, $\frac{1}{5n} < \frac{1}{5n-1}$.

• Since $\sum_{n=1}^{\infty} \frac{1}{5n} = \frac{1}{5}\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series, $\sum_{n=1}^{\infty} \frac{1}{5n-1}$ diverges by the comparison test.

4. The Limit Comparison Test

$a_n, b_n \geq 0$ for all n

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

Why?

5. Use the **limit comparison test** to determine whether the series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2 - \ln(n)}$

thinking: Looks close to conv. p-series $\sum \frac{1}{n^2}$.

5 points to make

Answer: • Compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$, convergent p-series.

• $\lim_{n\to\infty} \frac{\frac{1}{n^2-\ln(n)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-\ln(n)} \overset{H}{=} \lim_{n\to\infty} \frac{2n}{2n-\frac{1}{n}}$

$= \lim_{n\to\infty} \frac{2}{2-\frac{1}{n^2}} = \frac{2}{2} = 1. \neq 0.$

• So $\sum_{n=1}^{\infty} \frac{1}{n^2-\ln(n)}$ converges by limit comparison test.

(b) $\sum_{n=1}^{\infty} \frac{1}{\ln(n^{10}+n)}$

thinking: For large n, $n^{10}+n \approx n^{10}$. So $\ln(n^{10}+n) \approx 10\ln(n)$. And $\ln(n) < n$. So $\frac{1}{\ln n} > \frac{1}{n} \leftarrow$ terms in divergent p-series

Answer

• Compare to $\sum_{n=1}^{\infty} \frac{1}{n}$, a divergent p-series.

• $\lim_{n\to\infty} \frac{\frac{1}{\ln(n^{10}+n)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln(n^{10}+n)} \overset{H}{=} \lim_{n\to\infty} \frac{1}{\frac{10n^9}{n^{10}+1}}$

$= \lim_{n\to\infty} \frac{n^{10}+1}{10n^9} = \lim_{n\to\infty} \left(\frac{n}{10} + \frac{1}{10}\cdot\frac{1}{n^9}\right) = \infty$

• So $\sum_{n=1}^{\infty} \frac{1}{\ln(n^{10}+n)}$ diverges by the limit comparison test.

Why does the Limit Comparison Test work?

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0$, then for very large n, $\frac{a_n}{b_n} \approx c$

  or $a_n \approx c b_n$.

  So $\sum a_n \approx \sum c b_n = c \sum b_n$. So they converge or diverge together.

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then b_n grows much faster than a_n. So, if $\sum b_n$ converges, then $\sum a_n$ must converge.

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then a_n grows much faster than b_n. Thus if $\sum b_n$ diverges, $\sum a_n$ must also diverge.