Section 5.4: Comparison Tests Plus

For each series or test, provide a description of the series or statement of the test including what we know about convergence or divergence.

- geometric series

$$\sum_{n=1}^{\infty} ar^{n-1}, \quad |r|<1 \text{ converges}, \quad |r|\geq 1 \text{ diverges}$$

- p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}; \quad p>1 \text{ converges}, \quad p\leq 1 \text{ diverges}$$

- divergence test

Given $\sum_{n=1}^{\infty} a_n$

If $\lim_{n\to\infty} a_n = \begin{cases} c \neq 0 \\ \text{or} \\ \text{DNE} \end{cases}$, then $\sum a_n$ diverges

- integral test

Given $\sum a_n$. Find $f(x)$ that matches a_n on $n=1,2,3,\ldots$ and is decreasing. Then $\int_1^{\infty} f(x)\,dx$ + $\sum a_n$ converge/diverge together.

- comparison test

Given $\sum a_n$.

- To show $\sum a_n$ converges, find $\sum b_n$ that converges and $0 \leq a_n \leq b_n$.
- To show $\sum a_n$ diverges, find $\sum b_n$ that diverges and $0 \leq b_n \leq a_n$.

- limit comparison test

Given $\sum a_n$, $a_n, b_n \geq 0$

- Find $\sum b_n$ so that $\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0$. Then $\sum a_n$ and $\sum b_n$ converge or diverge together.
- Find convergent $\sum b_n$ so that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. Then $\sum a_n$ converges
- Find divergen $\sum b_n$ so that $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$. Then $\sum a_n$ diverges.

A. $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ Comparison Test. Pick $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$, a convergent geometric series.

Since $n2^n > 2^n$, $\frac{1}{n2^n} < \frac{1}{2^n}$. Since $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ converges, $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges.

B. $\sum_{n=1}^{\infty} 2^n$

Divergence Test

$\lim_{n\to\infty} 2^n = \infty \neq 0$. So $\sum_{n=1}^{\infty} 2^n$ diverges

C. $\sum_{n=1}^{\infty} \frac{n}{2^n}$. Limit Comparison Test. Compare to $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$, a convergent geometric series.

$$\lim_{n\to\infty} \frac{\frac{n}{2^n}}{\left(\frac{2}{3}\right)^n} = \lim_{n\to\infty} \frac{n}{2^n}\cdot\frac{3^n}{2^n} = \lim_{n\to\infty} \frac{n}{\left(\frac{4}{3}\right)^n} \overset{H}{=} \lim_{n\to\infty} \frac{1}{\ln\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)^n} = 0$$

So $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges.

D. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$. Integral Test.

Pick $f(x) = \frac{1}{x(\ln x)^3} = \frac{(\ln x)^{-3}}{x}$. Now $\int_2^{\infty} \frac{(\ln x)^{-3}}{x}\,dx = \lim_{b\to\infty} -\frac{1}{2}(\ln(x))^{-2}\Big]_2^b$

$$= \lim_{b\to\infty}\left(-\frac{1}{2}\left[\frac{1}{(\ln b)^2} - \frac{1}{(\ln 2)^2}\right]\right) = \frac{1}{2}(\ln 2)^{-2}.$$ So $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ converges.

E. $\sum_{n=1}^{\infty} \frac{n-4}{n^3+2n}$. limit Comparison Test. Pick $\sum_{n=1}^{\infty} \frac{1}{n^2}$, convergent p-series

$$\lim_{n\to\infty} \frac{\frac{n-4}{n^3+2n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3-4n^2}{n^3+2n} = 1 \neq 0. \text{ So } \sum_{n=1}^{\infty} \frac{n-4}{n^3+2n}$$

converges.

F. $\sum_{n=2}^{\infty} \frac{1+\cos(n)}{e^n}$. Comparison Test. Pick $\sum_{n=2}^{\infty} \frac{2}{e^n} = \sum_{n=2}^{\infty} 2\left(\frac{1}{e}\right)^n$, a convergent geometric series.

Since $0 \le 1+\cos(n) \le 2$, $\frac{1+\cos(n)}{e^n} \le \frac{2}{e^n}$.

So $\sum_{n=2}^{\infty} \frac{1+\cos(n)}{e^n}$ converges.

G. $\sum_{n=3}^{\infty} \frac{n^2}{\sqrt{n^3-1}}$. limit comparison test. Pick $\sum_{n=3}^{\infty} n^{1/2} = \sum_{n=3}^{\infty} \frac{1}{n^{-1/2}}$, a divergent p-series.

$$\text{So } \lim_{n\to\infty} \frac{\frac{n^2}{\sqrt{n^3-1}}}{\sqrt{n}} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3-1}} \cdot \frac{\frac{1}{n^{3/2}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1-\frac{1}{n^3}}} = 1 \neq 0$$

So $\sum \frac{n^2}{\sqrt{n^3-1}}$ diverges. [Should have done divergence test!]

H. $\sum_{n=1}^{\infty} \frac{n^3}{(n^4-3)^2}$ converges. Pick $\sum_{n=1}^{\infty} \frac{1}{n^5}$

Use limit comparison test

$$\lim_{n\to\infty} \frac{n^3}{(n^4-3)^2} \cdot \frac{n^5}{1} = \lim_{n\to\infty} \frac{n^8}{n^8-6n^4+6} = 1 \neq 0$$

I. $\sum_{n=1}^{\infty}(-1)^n 3^{-n/3}$ converges

$$(-1)^n 3^{-n/3} = \frac{(-1)^n}{3^{+n/3}} = \frac{(-1)^n}{(3^{1/3})^n} = \left(\frac{-1}{\sqrt[3]{3}}\right)^n$$

geometric with $\left|\frac{-1}{\sqrt[3]{3}}\right| < 1$.

J. $\sum_{n=2}^{\infty}\frac{1}{n!}$ converges.

for $n \geq 4$, $n! > n^2$. So $\frac{1}{n!} < \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series, $\sum \frac{1}{n!}$ converges.

n	1	2	3	4
n^2	1	4	9	16
$n!$	1	2	6	24

K. $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$. Integral Test Pick $f(x) = \frac{x}{x^2+1}$.

So $\int_1^{\infty}\frac{x\,dx}{x^2+1} = \lim_{b\to\infty} \frac{1}{2}\ln(x^2+1)\Big]_1^b = \lim_{b\to\infty}\frac{1}{2}\left(\ln(b^2+1) - \ln(2)\right)$

$= \infty$. So $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ diverges.

L. $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$. Converges. Limit comparison test.

Pick $\sum_{n=2}^{\infty}\frac{1}{n^2}$, a convergent p-series.

$$\lim_{n\to\infty}\frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^2}{n^2+1} = 1 \neq 0.$$