SECTION 5.5: ALTERNATING SERIES (DAY 2)

(1) The Alternating Series Test

Given $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ or $\sum_{n=1}^{\infty} (-1)^{n} b_n$, the series converges if:

(i) $0 < b_{n+1} \leq b_n$ for $n \geq 1$ and (ii) $\lim_{n\to\infty} b_n = 0$

(2) Determine whether the alternating series below converge or diverge. Justify your conclusion.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

$b_n = \frac{1}{n^2}$; (i) $0 < \frac{1}{(n+1)^2} \leq \frac{1}{n^2}$. So b_n decreases.

(ii) $\lim_{n\to\infty} \frac{1}{n^2} = 0$ Thus, the series converges by Alternating Series Test (AST)

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{3n+1}$

$b_n = \frac{n}{3n+1}$

$\lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3} \neq 0.$

So $\lim_{n\to\infty} \frac{(-1)^{n-1} \cdot n}{3n+1}$ does not exist.

So the series diverges by divergences test.

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2^n}$

$b_n = \frac{n}{2^n}$; (i) $0 < \frac{n+1}{2^{n+1}} \leq \frac{n}{2^n}$

Check: or $\frac{2^n}{2^{n+1}} \leq \frac{n}{n+1}$ or $\frac{1}{2} \leq \frac{n}{n+1}$ ok ✓

(ii) $\lim_{n\to\infty} \frac{n}{2^n} = 0$. So the series converges by AST.

(3) Remainders in Alternating Series and How to Estimate Them

Assume $S = \sum_{n=1}^{\infty} (-1)^{n+1} b_n$ is a convergent alt. series and S_N is the N^{th} partial sum. Then the remainder, R_N, is: $R_N = S - S_N$. and

- $|R_N| < b_{N+1}$.

(4) Recall that we concluded the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges. Find S_4 and determine how well it estimates the sum of the series. How large does k need to be so that S_k is within $0.0001 = 10^{-4}$ of the sum of the series?

- $S_4 = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{25} = 0.82\overline{1}$
- $|R_4| \leq b_6 = \frac{1}{36} = 0.02\overline{7}$

- Want $|R_N| < 10^{-4}$.

So $\frac{1}{N^2} < \frac{1}{10{,}000}$ or

$N^2 > 10{,}000$. So $\underline{\underline{N > 100}}$.

(5) **Definitions:** Absolute and Conditional Convergence

- If $\sum |a_n|$ converges, we say $\sum a_n$ is <u>absolutely convergent.</u>
- If $\sum a_n$ converges but $\sum |a_n|$ diverges, we say $\sum a_n$ is <u>conditionally convergent</u>
  - Standard Example of cond. conv series: $\sum (-1)^{n+1} \frac{1}{n}$!

(6) For each series below, determine if the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n^2}$ absolutely convergent

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{3n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{3n^2} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^2}$$, a multiple of a convergent p-series.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n+1}$ conditionally convergent.

- $\sum a_n$ converges by AST

Use $b_n = \frac{1}{3n+1}$, (i) decreasing. (ii) $\lim_{n \to \infty} \frac{1}{3n+1} = 0$ ✓

- $\sum |a_n| = \sum_{n=1}^{\infty} \frac{1}{3n+1}$ diverges by limit comparison test to $\sum \frac{1}{n}$.

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Show $\sum_{n=1}^{\infty} \frac{1}{3n+1}$ diverges by limit comparison to $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\frac{1}{3n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3}$$

Thus, since $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{3n+1}$ diverges.