SECTION 5.6: RATIO AND ROOT TESTS

(1) Recall the Ratio Test

Given $\sum_{n=1}^{\infty} a_n$. Find $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$ ← may be $+\infty$

- If $r < 1$, $\sum a_n$ converges
- If $r > 1$, $\sum a_n$ diverges
- If $r = 1$, no information.

Why? If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$, then $a_{n+1} \approx r a_n$, $a_{n+2} \approx r a_{n+1} \approx r^2 a_n$

So $a_{n+3} \approx r^3 a_n$, $a_{n+4} \approx r^4 a_n \ldots$ Looks geometric!

(2) Use the Ratio Test to determine if the series $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ converges or diverges, or explain why the test fails.

$$\lim_{n\to\infty} \frac{\frac{(2n+2)!}{((n+1)!)^2}}{\frac{(2n)!}{(n!)^2}} = \lim_{n\to\infty} \frac{(2n+2)!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = 4 > 1. \quad \text{So } \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} \text{ diverges.}$$

(3) The Root Test Given $\sum_{n=1}^{\infty} a_n$. Find $\lim_{n\to\infty} \sqrt[n]{|a_n|} = r$

- If $r < 1$, $\sum a_n$ converges
- If $r > 1$, $\sum a_n$ diverges
- If $r = 1$, no information.

→ why? $|a_n| \approx r^n$

(4) Use the Root Test on each series below to determine if it converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{(n+1)^{2n}}{(5n^2+n)^n}$

$$\lim_{n\to\infty} \sqrt[n]{\frac{(n+1)^{2n}}{(5n^2+n)^n}} = \lim_{n\to\infty} \frac{(n+1)^2}{5n^2+n} = \frac{1}{5} < 1$$

So $\sum_{n=1}^{\infty} \frac{(n+1)^{2n}}{(5n^2+n)^n}$ converges.

* Aside : $(a^p)^q = a^{pq} = (a^q)^p$

So $\left((n+1)^{2n}\right)^{\frac{1}{n}} = (n+1)^2$

* Aside : $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Why? $\lim_{n\to\infty} \frac{1}{n}\ln(n) = \lim_{n\to\infty} \frac{\ln(n)}{n} \overset{H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$. So $\lim_{n\to\infty} n^{\frac{1}{n}} = e^0 = 1$.

(b) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{\left(n^{1/n}\right)^2}{2} = \lim_{n\to\infty} \frac{1}{2} < 1$$

So $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

(5) Find the values of x for which the series $\sum_{k=1}^{\infty} \frac{x^k}{k^4}$ converges. Explain your answer.

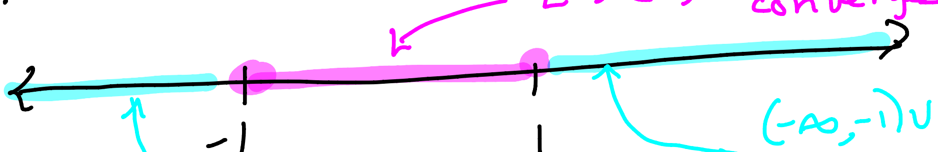
Apply Ratio Test

$$\lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)^4} \cdot \frac{k^4}{x^k} \right| = \lim_{k\to\infty} |x| \cdot \frac{k^4}{(k+1)^4} = |x| \cdot \lim_{k\to\infty} \frac{k^4}{(k+1)^4} = |x|$$

So if $|x| < 1$, the series converges. If $|x| > 1$, then the series diverges. If $x = 1$, $\sum \frac{1}{k^4}$ converges. If $x = -1$, $\sum \frac{(-1)^k}{k^4}$ converges.

ANS:

$[-1, 1]$, series converges



$(-\infty, -1) \cup (1, \infty)$ series diverges