(1) Principle: Provided the x-values of the power series are in the interval of convergence, then

we can operate term-by-term including addition, multiplication, differentiation, and integration.

(2) Use partial fractions to find a power series representation of $f(x) = \dfrac{1}{(x-1)(x-3)}$

$$f(x) = \frac{1}{(x-1)(x-3)} \underset{\text{partial fractions}}{=} \frac{-1/2}{x-1} + \frac{1/2}{x-3} = \frac{1}{2}\cdot\frac{1}{1-x} - \frac{1}{2}\frac{1}{3-x}$$

$$= \frac{1}{2}\cdot\frac{1}{1-x} - \frac{1}{6}\cdot\frac{1}{1-\frac{x}{3}} = \frac{1}{2}\sum_{n=0}^{\infty} x^n - \frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2}x^n - \frac{1}{6}\left(\frac{x}{3}\right)^n\right] = \sum_{n=0}^{\infty}\left(\frac{1}{2} - \frac{1}{6}\left(\frac{1}{3}\right)^n\right)x^n$$

I.o.C $(-1,1)$

I.o.C $(-3,3)$

pick smallest interval.

I.o.C. $(-1, 1)$

(3) Given $f(x) = \sum_{n=1}^{\infty} x^n$ and $g(x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n$, find the power series representation of $f(x) \cdot g(x)$.

$$f(x) = \sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + x^4 + \cdots$$

$\leftarrow$ I.o.C $(-1,1)$

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots$$

$\leftarrow$ I.o.C $[-1,1)$

$$f(x) \cdot g(x) = \left(x + x^2 + x^3 + x^4 + x^5 + \cdots\right)\left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \cdots\right)$$

$$= x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^4 + \frac{1}{4}x^5 + \frac{1}{5}x^6 + \cdots + x^3 + \frac{1}{2}x^4 + \frac{1}{3}x^5 + \frac{1}{4}x^6 + \frac{1}{5}x^7 + \cdots$$

$$+ x^4 + \frac{1}{2}x^5 + \frac{1}{3}x^6 + \frac{1}{4}x^7 + \frac{1}{5}x^8 + \cdots$$

$$+ x^5 + \frac{1}{2}x^6 + \frac{1}{3}x^7 + \cdots$$

$$= x^2 + \left(1 + \frac{1}{2}\right)x^3 + \left(1 + \frac{1}{2} + \frac{1}{3}\right)x^4 + \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)x^5 + \cdots$$

$\uparrow$ collect terms

$$= \sum_{n=2}^{\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}\right) x^n$$

(4) Differentiate the given series expansion of f term-by-term to obtain a series expansion for the derivative of f.

$$f(x) = \frac{2}{2-x} = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

F.Y.I. $\frac{2}{2-x} \cdot \frac{\frac{1}{2}}{\frac{1}{2}} = \frac{1}{1-\frac{x}{2}}$ and I.o.C is $(-2,2)$.

differentiate

$$f'(x) = 2(-1)(2-x)^{-2}(-1) = \frac{2}{(2-x)^2}$$ left-hand side

$$= \sum_{n=1}^{\infty} n\left(\frac{x}{2}\right)^{n-1} \cdot \left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{n}{2^n} x^{n-1}$$ right-hand side.

Now we have a power series expansion for a new function:

$$g(x) = \frac{2}{(2-x)^2} = \sum_{n=1}^{\infty} \frac{n}{2^n} x^{n-1}$$

I.o.C $(-2,2)$

So $g(1)$ is defined.

(5) Use your answer to the previous problem to determine the sum of the convergent series $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

So $$\sum_{n=1}^{\infty} \frac{n}{2^n} = g(1) = \frac{2}{(2-1)^2} = 2.$$

(6) Find a power series representation of $f(x) = \frac{1}{1+x^2}$ and integrate the series expansion term-by-term to obtain a series expansion for the indefinite integral of f.

$$f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Integrate both sides:

- L.h.s. $\arctan(x) = \int_0^x \frac{1}{1+t^2}\,dt = \arctan(t)\Big]_0^x$

$$= \arctan(x)$$

- R.h.s.

$$\int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^{2n}\,dt \right) = \int_0^x \left(1 - t^2 + t^4 - t^6 + \dots\right) dt$$

$$= \sum_{n=0}^{\infty} \left(\int_0^x (-1)^n t^{2n}\,dt \right) = \sum_{n=0}^{\infty} \left((-1)^n \cdot \frac{t^{2n+1}}{2n+1} \Big]_0^x \right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1};$$

So $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ I.o.c $(-1,1)$