Section 6.3: Taylor and Maclaurin Series (A first look)

why?

(1) Definitions

If $f(x)$ has derivatives of all orders at $x = a$, then the **Taylor series** for $f(x)$ at $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

The Taylor series where $a = 0$, is called the **Maclaurin series**:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

(2) Find the Taylor series for $y = \ln(x)$ at $x = 1$. at $x=1$

	at $x=1$
$f(x) = \ln(x)$	$f(1) = 0$
$f'(x) = x^{-1}$	$f'(1) = 1$
$f''(x) = -x^{-2}$	$f''(1) = -1$
$f'''(x) = 2x^{-3}$	$f'''(1) = 2$
$f^{(4)}(x) = -3\cdot 2\cdot 1\, x^{-4}$	$f^{(4)}(1) = -3!$
$f^{(5)}(x) = 4\cdot 3\cdot 2\cdot 1\, x^{-5}$	$f^{(5)}(1) = 4!$
$\vdots$	
$f^{(n)}(x) = (-1)^{n+1}(n-1)!\, x^{-n}$	$f^{(n)}(1) = (n-1)!\,(-1)^{n+1}$

So $$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!}\cdot(x-1)^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$$

Where do these formulas come from?

Suppose $f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \dots$

we want to determine the coefficients, c_i.

then

- $f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \dots + nc_n x^{n-1} + \dots$
- $f''(x) = 2c_2 + 3\cdot 2 c_3 x + 4\cdot 3 \cdot c_4 x^2 + 5\cdot 4 c_5 x^3 + \dots + n(n-1)c_n x^{n-2} + \dots$
- $f'''(x) = 3\cdot 2\cdot c_3 + 4\cdot 3\cdot 2 c_4 x + 5\cdot 4\cdot 3 c_5 x^2 + \dots + n(n-1)(n-2)c_n x^{n-3} + \dots$
- $f^{(4)}(x) = 4!\, c_4 + 5!\, c_5 x + \dots + n(n-1)(n-2)(n-3) c_n x^{n-4} + \dots$

So $f(0) = c_0$

$f'(0) = c_1$

$f''(0) = 2!\, c_2$ or $c_2 = \dfrac{f''(0)}{2!}$

$f'''(0) = 3!\, c_3$ or $c_3 = \dfrac{f'''(0)}{3!}$

$f^{(4)}(0) = 4!\, c_4$ or $c_4 = \dfrac{f^{(4)}(0)}{4!}$

$\vdots$

$f^{(n)}(0) = n!\, c_n$ or $c_n = \dfrac{f^{(n)}(0)}{n!}$