Section 6.3: Remainders, Section 6.4: Intro

(1) Review from Friday (You should do this from memory OR from your notes from Friday.)

(a) If $f(x)$ has derivatives of all orders at $x = a$, then the **Taylor series** for $f(x)$ at $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

(b) Write the Taylor series for $y = \ln(x)$ at $x = 1$ and plug $x = 2$ into that formula.

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n; \quad \ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

the alternating harmonic series !!

(c) Write the Taylor series for $f(x) = \cos(x)$ at $a = \pi/2$ and write $p_1(x)$ and $p_3(x)$, the first and third Taylor polynomials for $f(x)$.

- $$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(x - \frac{\pi}{2}\right)^{2n+1}$$
- $$p_1(x) = x - \frac{\pi}{2}$$
- $$p_3(x) = \left(x - \frac{\pi}{2}\right) - \frac{\left(x - \frac{\pi}{2}\right)^3}{3!}$$

(2) Using the definition, find the Taylor series for $g(x) = e^x$ centered at $a = 0$.

$$g(x) = e^x \qquad g^{(n)}(0) = 1$$
$$g'(x) = e^x$$
$$\vdots$$
$$g^{(n)}(x) = e^x$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

(3) A careful look at **remainders**, R_n, of Taylor series.

$$S = f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \underbrace{f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{p_n(x)} + \underbrace{\frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \dots}_{R_n}$$

$$= p_n(x) + R_n$$

So $S = p_n(x) + R_n$ or $S \approx p_n(x)$, w/ error $|R_n|$

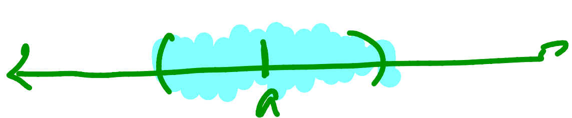
estimate sum w/ n^{th} partial sum

Remainder Thm:

$$|R_n| \le \frac{M}{(n+1)!}(x-a)^{n+1}$$

for x in interval I and

M = maximum value of $f^{(n+1)}(x)$ on I.



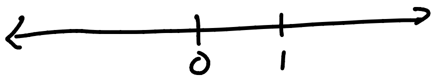
(4) Write $p_2(x)$, the 2nd Taylor polynomial for $g(x) = e^x$ centered at $x = 0$ and use it to estimate e. Estimate $|R_2|$ on the interval of the x-axis between the center ($a = 0$) and where we are estimating ($x = 1$).

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad ; \quad P_2(x) = 1 + x + \frac{1}{2}x^2$$

$$e = e^1 \approx P_2(1) = 1 + 1 + \frac{1}{2} \cdot 1^2 = 2.5$$

Estimate $|R_2|$ using Remainder Thm.

- $I = [0,1]$ or $[-1,1]$



- $|x-0|^{n+1} \le 1^{n+1} = 1$ on I

- $M = \max f^{(3)}(x)$ on I $= \max e^x$ on I $\le e^1 < 3$

So, $|R_2| \le \frac{3}{3!}(1) = \frac{1}{2}$

Bonus: How many terms to get w/i $1/100$ of correct value for e? Need $\frac{3}{(n+1)!} < \frac{1}{100}$ or $300 < (n+1)!$. $n=5$ is sufficient!

(5) Use the Taylor series from (2) on this sheet to find (quickly), the Taylor series for $g(x) = e^{x/2}$ and $h(x) = e^{x^2}$.

$$g(x) = e^{x/2} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^n n!} \leftarrow \text{Same thing we got last Fri!}$$

$$h(x) = e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!}(x^2)^n = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \underset{n=0}{1} + \underset{n=1}{x^2} + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \cdots$$

(6) Observe that the Taylor series for $h(x)$ allows us to solve a very hard problem!

$$\int_0^1 e^{x^2}\,dx = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\right) dx = \sum_{n=0}^{\infty} \left(\int_0^1 \frac{1}{n!} x^{2n}\,dx\right)\Bigg|_0^1$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n!} \cdot \frac{1}{(2n+1)} \cdot x^{2n+1} \Big]_0^1 \right) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)n!} = 1 + \frac{1}{3} + \frac{1}{5 \cdot 2!} + \frac{1}{7 \cdot 3!} + \cdots$$

In fact: $S_5 = 1.46253$ w/i $\frac{1}{10{,}000}$th of exact answer.