SECTION 6.3: TAYLOR AND MACLAURIN SERIES (DAY 2)

(1) Recall from previous day:

If $f(x)$ has derivatives of all orders at $x = a$, then the **Taylor series** for $f(x)$ at $x = a$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(2) Recall the Taylor series for $y = \ln(x)$ at $x = 1$ is:

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

cool stuff at end $\longrightarrow$

(3) Find the Taylor series for each function $f(x)$ at the given center $x = a$. (If you want to be ambitious, find their intervals of convergence!)

(a) $f(x) = \cos(x)$ at $a = \pi/2$

$f(x) = \cos(x)$

$f'(x) = -\sin(x) = f^{(5)}(x)$

$f''(x) = -\cos(x) = f^{(6)}(x)$

$f'''(x) = \sin(x) = f^{(7)}(x)$

$f^{(4)}(x) = \cos(x) = f(x)$

at $x = \pi/2$

$f(\pi/2) = 0 = f^{(4)}(\pi/2)$

$f'(\pi/2) = -1 = f^{(5)}(\pi/2)$

$f''(\pi/2) = 0 = f^{(6)}(\pi/2)$

$f'''(\pi/2) = 1 = f^{(7)}(\pi/2)$

Observations

- Only odd derivatives appear
- They alternate between 1 and -1

Answer: $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}\left(x - \pi/2\right)^{2n+1}$

(b) $f(x) = e^{x/2}$ at $x = 0$

$f(x) = e^{x/2}$

$f'(x) = \frac{1}{2}e^{x/2}$

$f''(x) = \left(\frac{1}{2}\right)^2 e^{x/2}$

$f'''(x) = \left(\frac{1}{2}\right)^3 e^{x/2}$

$\vdots$

$f^{(n)}(x) = \left(\frac{1}{2}\right)^n e^{x/2}$

$f^{(n)}(0) = \left(\frac{1}{2}\right)^n e^{0/2} = \left(\frac{1}{2}\right)^n.$

$$e^{x/2} = \sum_{n=0}^{\infty} \frac{1}{2^n n!} x^n$$

I.o.c $(-\infty, \infty)$

(4) Definition: The n-th Taylor *polynomial* of $f(x)$ centered at $x = a$ is:

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

(a) Find the first four Taylor polynomials, $p_0(x), p_1(x), p_2(x), p_3(x)$, for $f(x) = e^{x/2}$.

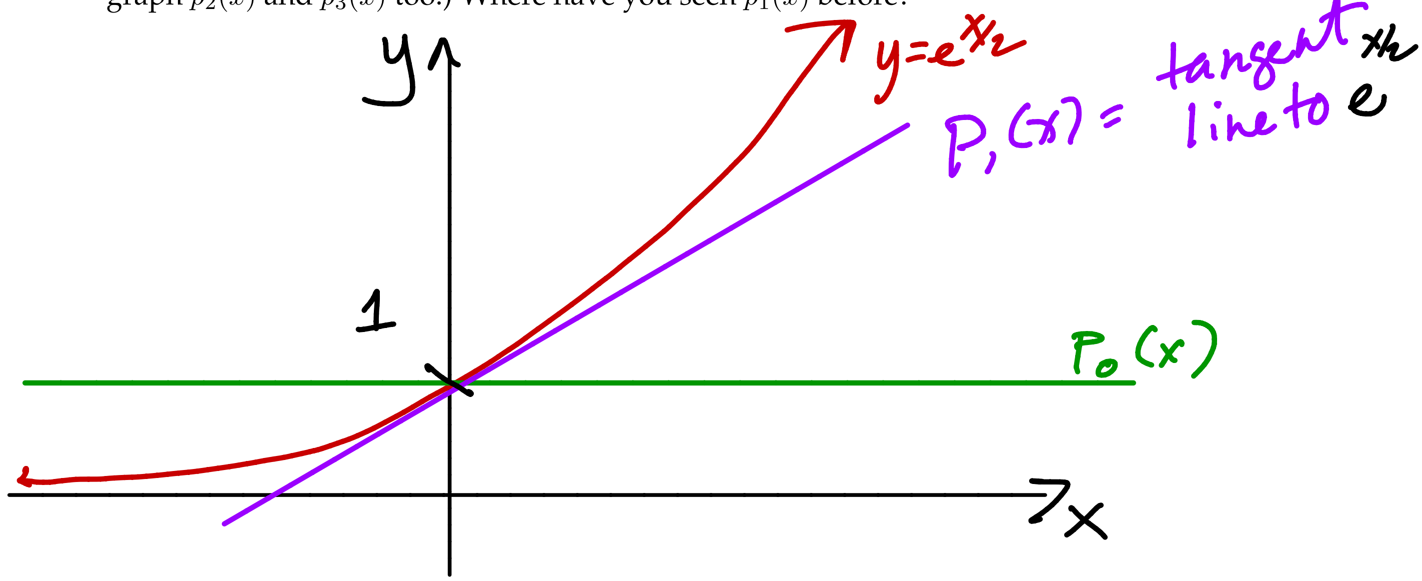
$$e^{x/2} = \sum_{n=0}^{\infty} \frac{1}{2^n n!} x^n = 1 + \frac{x}{2} + \frac{x^2}{2^2 \cdot 2!} + \frac{x^3}{2^3 \cdot 3!} + \cdots$$

- $P_0(x) = 1$
- $P_1(x) = 1 + \frac{x}{2}$

$P_2(x) = 1 + \frac{x}{2} + \frac{x^2}{8}$

$P_3(x) = 1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48}$

(b) Graph at least $f(x)$, $p_0(x)$ and $p_1(x)$ on the same set of axes. (If you want to be ambitious, graph $p_2(x)$ and $p_3(x)$ too.) Where have you seen $p_1(x)$ before?



$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n$$

I.O.C. : Use ratio test.

$$\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{n+1} \cdot \frac{n}{(x-1)^n} \right| = \lim_{n\to\infty} |x-1| \left(\frac{n}{n+1}\right) = |x-1| < 1$$

So $-1 < x-1 < 1$ or $0 < x < 2$.

At $x=2$? $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ $\leftarrow$ alt. harmonic series.

So it converges.

What cool thing did we learn??

The alternating harmonic series converges to $\ln(2)$. (!!)

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$