Section 6.4: Working With Taylor Series

(1) Write the Taylor Series of $f(x) = e^x$ at $a = 0$ either from memory or using the formula. State the interval of convergence.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{I.o.c. } (-\infty, \infty)$$

(2) **The Point of this Section and Chapter 5 and 6:**

We can solve hard problems we could not without power series.

(3) Evaluate $\displaystyle\int_0^1 e^{x^2}\,dx$. ← Not possible using techniques from Calc I or Ch 3 from Calc II.

- Write e^{x^2} as a power series: $\displaystyle e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

- Integrate the power series:

$$F(x) = \int_0^x e^{t^2}\,dt = \int_0^x \left(\sum_{n=0}^{\infty} \frac{t^{2n}}{n!}\,dt\right) = \sum_{n=0}^{\infty} \left(\int_0^x \frac{t^{2n}}{n!}\,dt\right) = \sum_{n=0}^{\infty} \left(\frac{t^{2n+1}}{(2n+1)n!}\right]_0^x$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!} = \underbrace{x}_{n=0} + \underbrace{\frac{x^3}{3}}_{n=1} + \underbrace{\frac{x^5}{5\cdot 2!}}_{n=2} + \underbrace{\frac{x^7}{7\cdot 3!}}_{n=3} + \underbrace{\frac{x^9}{9\cdot 4!}}_{n=4} + \cdots$$

$$F(1) = \int_0^1 e^{x^2}\,dx = \sum_{n=0}^{\infty} \frac{1}{(2n+1)n!} \approx \sum_{n=0}^{7} \frac{1}{(2n+1)n!} = 1.46265$$

w/i 0.000001 of exact answer.

(4) (a) $f(x)=\sqrt{x+1} = (x+1)^{\frac{1}{2}}$

• Find $f'''(0)$.

$f'(x) = \frac{1}{2}(x+1)^{-1/2}$

$f''(x) = \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)(x+1)^{-3/2}$

$f'''(x) = \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)(x+1)^{-5/2}$

$f'''(0) = \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right) = \binom{1/2}{3}$

$$f(x) = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n = \binom{\frac{1}{2}}{0}x^0 + \binom{\frac{1}{2}}{1}x^1 + \binom{\frac{1}{2}}{2}x^2 + $$

$$= 1 + \frac{1}{2}x + \frac{1}{2}\left(\frac{-1}{2}\right)\cdot\frac{1}{2!}x^2 + \ldots$$

(b) $g(x) = \frac{1}{\sqrt[3]{1+x}} = (1+x)^{-1/3}$

• Find $g^{(4)}(0)$

$g'(x) = -\frac{1}{3}(1+x)^{-4/3}$

$g''(x) = \left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)(1+x)^{-7/3}$

$g'''(x) = \left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)\left(\frac{-7}{3}\right)(1+x)^{-10/3}$

$g^{(4)}(x) = \left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)\left(\frac{-7}{3}\right)\left(\frac{-10}{3}\right)(1+x)^{-13/3}$

$g^{(4)}(0) = \left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)\left(\frac{-7}{3}\right)\left(\frac{-10}{3}\right) = \binom{-1/3}{4}$

$$g(x) = \sum_{n=0}^{\infty} \binom{-1/3}{n} x^n$$

(5) Definition of $\binom{r}{n} = \frac{r!}{n!(r-n)!} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!} = \boxed{\frac{r(r-1)(r-2)\cdots(r-(n-1))}{n!}}$

Practice: $\binom{7}{3} = \frac{7!}{3!\,4!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{3!\,4!} = \frac{7\cdot6\cdot5}{3!}$

$\binom{\frac{1}{2}}{3} = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}$ $n=3$

$= \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(-\frac{3}{2}\right)}{1}$

• $\binom{-\frac{1}{3}}{4} = \frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}-1\right)\left(\frac{-1}{3}-2\right)\left(\frac{-1}{3}-3\right)}{4!}$

$= \left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)\left(\frac{-7}{3}\right)\left(\frac{-10}{3}\right)/4!$

(6) The Taylor Series for $f(x) = (1+x)^r$.

$$\sum_{n=0}^{\infty} \binom{r}{n} x^n$$

(7) This is a step-by-step walk through problem # 234 from Section 6.4.

(a) Find the Taylor Series for $f(x) = \sin(x)$ at $a = 0$.

$f(x) = \sin(x)$

$f'(x) = \cos(x)$

$f''(x) = -\sin(x)$

$f'''(x) = -\cos(x)$

$f^{(4)}(x) = \sin(x)$

$f^{(5)}(x) = \cos(x)$

at $x=0$

$f(0) = 0 = f^{(4)}(0)$

$f'(0) = 1 = f^{(5)}(0)$

$f''(0) = 0$ $\quad\vdots$

$f'''(0) = -1$

Observe: Only odd terms appear. They alternate between +1 and -1.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

(b) Use the previous part to find the Taylor series for $f(x) = \sin(2x)$ at $a = 0$.

$$\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$$

(c) Show that $\sin^2(x) = \int_0^x 2\sin(t)\cos(t)\,dt$.

$$\int_0^x 2\sin(t)\cos(t)\,dt = 2\int_0^{\sin(t)} u \cdot du = u^2\Big]_0^{\sin(t)} = \sin^2(t)$$

Let $u = \sin(t)$
$du = \cos(t)\,dt$

(d) (# 243) Use the fact that $\sin(2x) = 2\sin(x)\cos(x)$ to find a power series representation for $\sin^2(x)$

$$\sin^2(x) = \int_0^x 2\sin(t)\cos(t)\,dt = \int_0^x \sin(2t)\,dt = \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} t^{2n+1}}{(2n+1)!}\right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{(-1)^n 2^{2n+1}}{(2n+1)!} \int_0^x t^{2n+1}\,dt\right) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} \cdot \frac{t^{2n+2}}{2n+2} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+2)!} t^{2n+2}$$