(1) Translating Calculus Ideas to Parametric Curves

Suppose you are given a curve defined as $x(t)$ and $y(t)$:

(a) $\frac{dy}{dx}$ $= \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

(b) $\frac{d^2y}{dx^2}$ $= \dfrac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$

(c) area under curve

$$A = \int_a^b f(x)\,dx = \int_a^b y\,dx = \int_{t=\alpha}^{t=\beta} y(t)\cdot x'(t)\,dt$$

(d) arc length

$$L = \int_{t=\alpha}^{t=\beta} \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

See last page for explanation

(2) Given the parametric equations $x(t) = t^3 + 1$, $y(t) = 2t - t^2$, answer the following questions without eliminating the parameter.

(a) Find dy/dx and d^2y/dx^2.

$$\frac{dx}{dt} = 3t^2,\ \frac{dy}{dt} = 2-2t;\ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2-2t}{3t^2} = \boxed{\frac{2}{3}\left(t^{-2} - t^{-1}\right)}$$

$$\frac{d}{dt}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left[\frac{2}{3}\left(t^{-2} - t^{-1}\right)\right] = \frac{2}{3}\left(-2t^{-3} + t^{-2}\right)$$

$$\boxed{\frac{d^2y}{dx^2}} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{2}{3}\left(-2t^{-3}+t^{-2}\right)}{3t^2} = \boxed{\frac{2}{9}\left(-2t^{-5} + t^{-4}\right)}$$

$x(t) = t^3+1, \quad y(t) = 2t - t^2$

(b) Write the equation of the tangent line to the curve at $t = 1$.

$x(1) = 2, \; y(1) = 2-1 = 1$ point $(2,1)$

$\boxed{y = 1}$

Slope $\left.\frac{dy}{dx}\right|_{t=1} = \frac{2}{3}\left(\frac{1}{1^2} - \frac{1}{1}\right) = 0$

(c) Is the curve concave up or concave down at $t = 1$?

$$\left.\frac{d^2y}{dx^2}\right|_{t=1} = \frac{2}{9}\left(\frac{-2}{1^5} + \frac{1}{1^4}\right) = \frac{2}{9}(-1) = \frac{-2}{9} < 0$$

$\boxed{\text{concave down}}$

(d) Determine the area below the curve and above the x-axis.

$y = 0$ when $t = 0$ and $t = 2$.

$$A = \int_0^2 y\,dx = \int_0^2 (2t - t^2)(3t^2)\,dt = \int_0^2 (6t^3 - 3t^4)\,dt$$

$$= \frac{6}{4}t^4 - \frac{3}{5}t^5\Big]_0^2 = \frac{6}{4}2^4 - \frac{3}{5}t^5 = 24 - \frac{96}{5} = \frac{24}{5}$$

(3) Determine the arc length of the cycloid $x(\theta) = \theta - \sin(\theta)$ and $y(\theta) = 1 - \cos(\theta)$ from $t = 0$ to $t = 2\pi$.

- $\frac{dx}{d\theta} = 1 - \cos(\theta), \; \frac{dy}{d\theta} = \sin\theta$
- $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (1 - \cos\theta)^2 + \sin^2\theta$

$= 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta$

$= 2 - 2\cos\theta.$

- $\sqrt{2 - 2\cos\theta} = \sqrt{4\left(\frac{1}{2} - \frac{1}{2}\cos\left(2\left(\frac{\theta}{2}\right)\right)\right)}$

$= 2\sqrt{\sin^2\left(\frac{\theta}{2}\right)} = 2\left|\sin\left(\frac{\theta}{2}\right)\right|$

use: $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$

- $L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$= 2\int_0^{2\pi} \left|\sin\left(\frac{\theta}{2}\right)\right| d\theta$

$= 2\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta = -4\cos\left(\frac{\theta}{2}\right)\Big]_0^{2\pi}$

$= -4\left(\cos(\pi) - \cos(0)\right) = -4(-1-1)$

$= 8$

1.a

Chain Rule: if $z = f(w)$, $w = g(v)$

then $\dfrac{dz}{dv} = \dfrac{dz}{dw} \cdot \dfrac{dw}{dv}$

Sppose $y = f(x)$, $x = g(t)$, then

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

or $\boxed{\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}}$

1b Suppose we have $\dfrac{dy}{dx}$ and want $\dfrac{d^2y}{dx^2}$,

then use $\dfrac{d}{dx}\left[\dfrac{dy}{dx}\right] = \boxed{\dfrac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}}$

1c If $y \geq 0$ and $x' \geq 0$ on $t = \alpha$, to $t = \beta$,

then area between y and x-axis is

$$A = \int_a^b y \, dx = \int_{t=\alpha}^{t=\beta} y(t)\, x'(t)\, dt$$

1d $$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{t=\alpha}^{t=\beta} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\ \frac{dx}{dt} \cdot dt$$

$$= \int_{t=\alpha}^{t=\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$