SOLUTIONS

Things to know by the end of this section

a. how to use sigma notation *with facility*

b. the meaning of a *series*, especially as compared to a *sequence* (from §5.1)

c. the meaning of *a sequence of partial sums of a series* and how to find it.

d. what it means to say a series converges.

e. what a *geometric series* is and how to determine whether or not it converges.

f. what a *telescoping series* is and how to determine whether or not it converges.

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1. An infinite series is

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

The sequence of its partial sums is

$$\left.\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_k &= a_1 + a_2 + \dots + a_k \end{aligned}\right\} \quad S_k = \sum_{n=1}^{k} a_n$$

2. For each series below, expand the sigma notation and then *compute and simplify the first 4 partial sums* S_1, S_2, S_3, S_4. (Use a calculating device to get a decimal, if desired.)

← I don't desire...

(a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots$

$S_1 = \frac{2}{3}$, $S_2 = \frac{2}{3} + \frac{4}{9} = \frac{10}{9}$, $S_3 = \frac{10}{9} + \frac{8}{27} = \frac{38}{27}$,

$S_4 = \frac{38}{27} + \frac{16}{81} = \frac{130}{81}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \dots$

$S_1 = \frac{1}{2}$, $S_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, $S_3 = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$,

$S_4 = \frac{3}{4} + \frac{1}{20} = \frac{4}{5}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{5^n} = -\frac{1}{5} + \frac{1}{5^2} - \frac{1}{5^3} + \frac{1}{5^4} - \dots$

$S_1 = -\frac{1}{5}$, $S_2 = -\frac{1}{5} + \frac{1}{25} = -\frac{4}{25}$,

$S_3 = -\frac{4}{25} - \frac{1}{125} = \frac{-21}{125}$, $S_4 = \frac{-21}{125} + \frac{1}{625} = \frac{-104}{625}$

(d) $\sum_{n=1}^{\infty} \frac{n}{n^2+2} = \frac{1}{1+2} + \frac{2}{4+2} + \frac{3}{9+2} + \frac{4}{16+2} + \cdots$

$S_1 = \frac{1}{3}$, $S_2 = \frac{1}{3} + \frac{2}{6} = \frac{2}{3}$, $S_3 = \frac{2}{3} + \frac{3}{11} = \frac{31}{33}$,

$S_4 = \frac{31}{33} + \frac{4}{18} = \frac{31}{33} + \frac{2}{9} = \frac{115}{99}$

3. *Complete the bulleted lines below.* **Definition:** The series $\sum_{n=1}^{\infty} a_n$

- converges if $\lim_{k\to\infty} S_k = \lim_{k\to\infty} \sum_{n=1}^{k} a_n$ exists and is finite
- diverges if $\lim_{k\to\infty} S_k$ does not exist or equals $\pm\infty$

4. The series in 2(a) is a geometric series. Its kth partial sum is $S_k = \sum_{n=1}^{k} \left(\frac{2}{3}\right)^n$. Below, write out S_k without sigma notation, multiply by $2/3$, then subtract, and then cancel as many terms as possible:

subtract

$$S_k = \frac{2}{3} + \cancel{\left(\frac{2}{3}\right)^2} + \cancel{\left(\frac{2}{3}\right)^3} + \cancel{\cdots} + \cancel{\left(\frac{2}{3}\right)^k}$$

$$\frac{2}{3}S_k = \cancel{\left(\frac{2}{3}\right)^2} + \cancel{\left(\frac{2}{3}\right)^3} + \cdots + \cancel{\left(\frac{2}{3}\right)^k} + \left(\frac{2}{3}\right)^{k+1}$$

$$\left(1 - \frac{2}{3}\right) S_k = \frac{1}{3} S_k = \frac{2}{3} - \left(\frac{2}{3}\right)^{k+1}$$

This has led to a closed formula for S_k:

$$S_k = 3\left(\frac{2}{3} - \left(\frac{2}{3}\right)^{k+1}\right)$$

$= 3\left(\frac{2}{3} - 0\right) = 2$

Therefore the infinite series converges to $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \lim_{k\to\infty} S_k = \boxed{2}$.

5. Do the same for the geometric series in 2(c).

$$S_k = -\frac{1}{5} + \cancel{\left(\frac{-1}{5}\right)^2} + \cancel{\cdots} + \cancel{\left(\frac{-1}{5}\right)^k}$$

$$-\frac{1}{5} S_k = \cancel{\left(\frac{-1}{5}\right)^2} + \cancel{\cdots} + \cancel{\left(\frac{-1}{5}\right)^k} + \left(\frac{-1}{5}\right)^{k+1}$$

$$\underbrace{\left(1 - \left(\frac{-1}{5}\right)\right)}_{=6/5} S_k = -\frac{1}{5} - \left(\frac{-1}{5}\right)^{k+1}$$

$\therefore$ $S_k = \frac{5}{6}\left(-\frac{1}{5} - \left(\frac{-1}{5}\right)^{k+1}\right)$ $\therefore$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{5^n} = \frac{5}{6}\left(-\frac{1}{5} - 0\right) = \boxed{-\frac{1}{6}}$