SOLUTIONS

# SECTION 5.3: DIVERGENCE AND INTEGRAL TESTS

1. The Divergence Test:

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n = \left(\text{anything other than zero}\right)$

2. The Integral Test: for $\sum_{n=1}^{\infty} a_n$, assuming $a_n \geq 0$, $a_n$ decreases, $a_n = f(n)$ for $f(x)$ continuous & decreasing,

$\boxed{\sum_{n=1}^{\infty} a_n}$ $\boxed{\int_N^{\infty} f(x)\,dx}$

both converge or both diverge

3. For each series below, find the limit if the *terms* of the series and determine **if** the Divergence Test applies. If the test applies, draw a conclusion.

   (a) $\sum_{n=1}^{\infty} \frac{n}{40n+30}$

   $\lim_{n\to\infty} \frac{n}{40n+30} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{n\to\infty} \frac{1}{40} = \frac{1}{40}$

   $\therefore$ series diverges

   (b) $\sum_{n=1}^{\infty} \frac{n}{40n^2+30}$

   $\lim_{n\to\infty} \frac{n}{40n^2+30} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{80n} = 0$

   no conclusion (test does not apply)

   (c) $\sum_{n=1}^{\infty} 8^{(n^{-2})}$

   $\lim_{n\to\infty} 8^{(n^{-2})} = 8^{\left(\lim_{n\to\infty} 1/n^2\right)} = 8^0 = 1$

   $\therefore$ series diverges

4. Why is the following claim FALSE?: "The series $\sum_{n=1}^{\infty} a_n$ converges because $a_n \to 0$ as $n \to \infty$."

① by example, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but $\lim_{n\to\infty} \frac{1}{n} = 0$

② generally, because $a_n \to 0$ but perhaps not fast enough!

5. Apply the integral test to $\sum_{n=1}^{\infty} \frac{1}{n^p}$, assuming $p > 1$.

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \lim_{t\to\infty}\left[\frac{t^{1-p}}{1-p} - \frac{1}{1-p}\right]$$

$$= 0 + \frac{1}{p-1} = \frac{1}{p-1} \quad \therefore \text{ series } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges}$$

$1-p<0$

6. Apply the integral test to $\sum_{n=1}^{\infty} \frac{1}{n^p}$, assuming $0 < p \leq 1$.

$p=1$: $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} [\ln x]_1^t = \lim_{t\to\infty} \ln t = +\infty \quad \therefore$ diverges

$0<p<1$: $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p} = +\infty$

$1-p>0$

$\therefore$ diverges

7. $p$**-series convergence**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1 \text{ and diverges if } p \leq 1.$$

Apply the above rule about $p$-series to determine whether the series below converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^{1.56}}$ $\quad p = 1.56 > 1 \quad \therefore$ converges

(b) $\sum_{n=1}^{\infty} \frac{1}{n^{99/100}}$ $\quad p = \frac{99}{100} < 1 \quad \therefore$ diverges